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Design of Linear Functional Noncircular Gear with High Contact Ratio Used in Continuously Variable Transmission

Yanan Hu[1,2], Chao Lin[1*], Chunjiang He[1], Yongquan Yu[1] and Zhiqin Cai[3]

Abstract

Continuously variable transmission (CVT) of noncircular gear has the technical advantages of large bearing capacity and high transmission efficiency. The key technology of CVT with noncircular gear has been broken through some countries, and is in the stage of deep application research. Although the characteristics and design methods of noncircular gear pairs have been continuously studied in China, the noncircular gear CVT is still in the preliminary exploration and research stage. The linear functional noncircular gear pair, whose transmission ratio is a linear function in the working section, to realize continuously variable transmission was the research object in this paper. According to the required transmission ratio in the working section, the transmission ratio function in the non-working section was constructed by using a polynomial. And then the influence of pitch curve parameters in the working section on which in the non-working section was also analyzed to obtain the pitch curve suitable for transmission of this gear pair. In addition, for improving the stability and bearing capacity of gear transmission, the noncircular gear pair transmission with high contact ratio was designed. Furthermore, the accurate value of the contact tooth length was calculated based on the gear principle and the characteristics of the involute tooth profile, from this the contact tooth length error was calculated by comparing the accurate value with its actual value obtained by the rolling experiment. Finally, an indirect method to verify the contact ratio by detecting the contact length error of the tooth profile was proposed.

Keywords Continuously variable transmission, Noncircular gear, Pitch curve, Contact ratio, Contact tooth profile

1 Introduction

Although continuously variable transmission (CVT) technology has been used in vehicles for only several decades, its advantages over traditional transmission are obvious: it has a wider range of working speed ratio; and it is easier to form an ideal match with the engine, so as to improve the combustion process, and further reduce fuel consumption and emissions; it has higher transmission efficiency, less power loss, and higher economy [1]. Generally, three main transmission modes can be adopted to realize infinitely variable speeds, namely, liquid transmission, electric transmission, and mechanical transmission. Different from the liquid transmission mode with a higher sliding rate, and the electric transmission mode with lower efficiency and unstable operation at low speed, in this respect, mechanical continuously variable transmission can easily realize effective transmission under special work conditions, such as high load, due to the equipment of more compact structure [2].

Furthermore, the transmission mechanism of mechanical CVT mainly includes belt transmission

*Correspondence:
Chao Lin
linchao@cqu.edu.cn
[1] State Key Laboratory of Mechanical Transmission, Chongqing University, Chongqing 400044, China
[2] College of Mechanical and Electrical Engineering, Binzhou University, Binzhou 256600, China
[3] School of Aeronautics and Astronautics, Xiamen University, Xiamen 361005, China

[3], chain transmission [4], and gear transmission [5]. Among them, belt CVT and chain CVT are the most common CVTs used in vehicles due to their simple structure, small size, and lightweight [6]. Despite these advantages, their limited torque capacity and low transmission efficiency cannot be ignored.

With the development of the design and manufacturing technology of noncircular gear, combined with the characteristics of noncircular gear transmission with non-uniform speed ratio, which belongs to conjugate meshing transmission [7–9], continuously variable transmission can be realized by using the transmission characteristics of noncircular gear, which is expected to improve the power and torque transmitted [10].

Many scholars have done a lot of research on the transmission technology of noncircular gears to solve many problems gradually, such as the geometric design of pitch curves, and the configuration and processing of gear teeth. Recently, the transmission technology of noncircular gear pairs has entered a new practical period [11]. Emura and Arakawa [12] utilized a noncircular gear for steering mechanism analysis. Song et al. [13] put forward that the gear-shifting process structure using a noncircular gear drive instead of a friction drive. Detailed descriptions of the gear trains for tracking the motions of the Mercury planet and of the Moon are herein presented by Addomine et al. [14]. In these gear trains, noncircular gears have been used to account for the apparent irregular motion of the planets. In addition, some scholars also use noncircular gear transmission in continuously variable transmission and have achieved some good results. Dooner et al. [15] recommended a continuously variable device of noncircular gears with a serrated transmission ratio. CVT could be realized within a specified range when controllable phase shift and planetary additive differential were used between noncircular gear devices [16].

For one noncircular gear pair, the shape and closure of the pitch curve have a decisive influence on the motion requirements. The pitch curves also have sharp points that require complex modification but can be continuous if the design is reasonable. Among them, the range of transmission ratio of the adder is limited. While the subtractor has a power cycle, its range of transmission ratio is arbitrary [17]. The pitch curves of noncircular gears were identified to be closed by the residue theorem [18]. Moreover, a general mathematical model for designing the pitch curve with minimal rotary inertia of the noncircular gear was proposed by resorting to the kinematics principle and calculus of variations [19]. And to realize a specific variable transmission ratio mechanism, a set of noncircular planetary gears consisting of an external pair and an internal pair with variable pitch lines was designed [20].

As for the contact ratio of gear transmission, it is closely related to mesh stiffness, which is an important parameter of gear transmission design and analysis [21]. As an index to measure the bearing capacity and transmission smoothness of gears, the higher the contact ratio is, the better the transmission smoothness is [9].

In other ways, Zheng et al. [22] put forward a new application of noncircular gears, which is an indexing mechanism using noncircular gears. He et al. [23] also provided a new generation method for noncircular gear, through which the generated gear may embody the advantages of localized tooth contact and good lubrication in practice. Based on the gear with rack and geared segment, Alexandru et al. [24] presented the geometric and functional characteristics of a gear with ascending variable ratio. This gear can be used for the steering boxes of some low-power vehicles (without a servo system).

Starting with the required transmission ratio, the influence of linear functional working pitch curve parameters on the non-working pitch curve was mainly studied in this paper. And the parameters of the linear functional noncircular gear pair suitable for transmission were obtained. By the way, an indirect method to verify the contact ratio by detecting the contact length error of the tooth profile was proposed, and the high contact ratio of the noncircular gear pair was verified. The research in this paper will contribute to reducing the transmission chain and simplifying the noncircular gear design used for CVT.

2 Design of Pitch Curve of Noncircular Gear in CVT

2.1 Realization of Infinitely Variable Transmission Using Noncircular Gears

The constant transformation from the input shaft to the output shaft can be realized by continuously variable noncircular gear transmission through the continuous and alternating transmission of power in the range of 0 to 360° by the multi-branch loop to obtain a constant speed ratio. Each branch loop is a three-element planetary row for differential coupling of two-order parallel noncircular gear pairs with non-uniform output speed, and then the constant speed ratio within a certain angle range is obtained through the controllable overrunning clutch. Rotation shafts of this branch are free rotation without power transmission, ranging in another rotation angle. Accordingly, the speed ratio can be continuously adjusted by changing the phase angle of two parallel noncircular gear pairs, like gear 1 and gear 2.

The continuously variable noncircular gear transmission system model and skeleton of the scheme

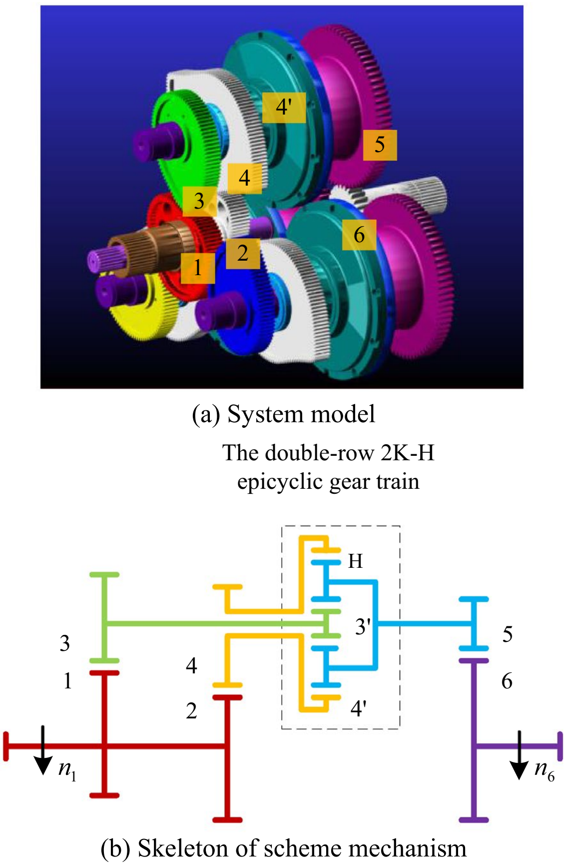


Figure 1 Continuously variable noncircular gear transmission

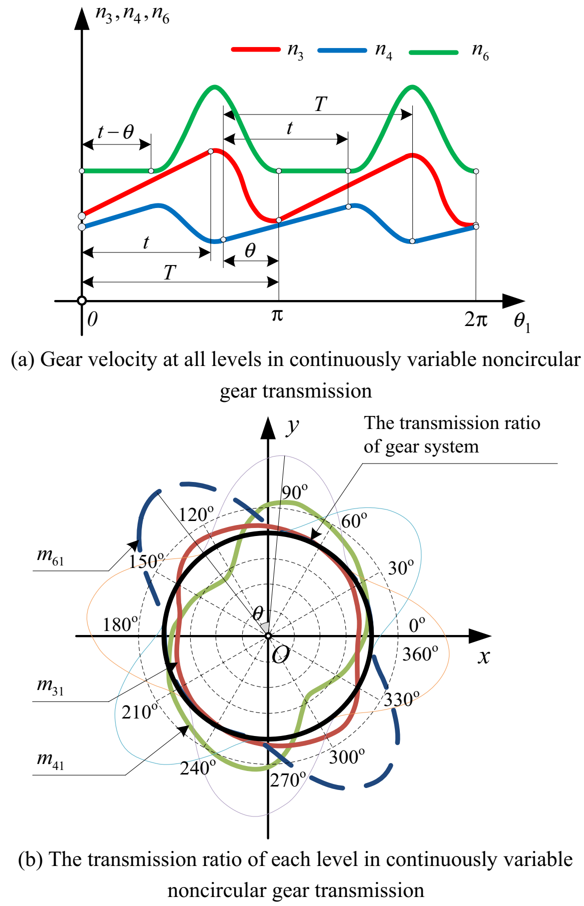


Figure 2 The basic transmission characteristics of continuously variable transmission with noncircular gear

mechanism are shown in Figure 1. As shown in Figure 1, 1–3 and 2–4 gear pairs are two noncircular gear pairs, while the rest are cylindrical gear pairs. This system with a constant transmission ratio involves the transmission unit superposition of two branch noncircular gear pairs with variable phases. During the working process, a constant speed input of component 1 will be converted to the linear function output speed of component 3 and component 4, through the first level transmission of the noncircular gear pair. And the constant output speed can be obtained in a limited range of angles after the output speed of components 3 and 4 is differential coupling. It is noted that the single-branch noncircular gear pair unit can only achieve stable speed at a limited angle. Once the three branches with second-order noncircular gear pairs are properly arranged around the central axis, and the minimum working angle of each branch per cycle is 120°, as a result, constant speed output will be obtained during the random angle range.

The output rotational speed of the noncircular gear pair designed by the pitch curve is shown in Figure 2(a). Namely, the single noncircular gear pair can't provide a steady output rotational speed. And then, the differential mechanism, that is the double-row 2K-H type epicyclical gear train with a specific number of teeth shown in Figure 1, can realize the constant synthesis speed and the stable output within the limited angle. Therefore, the design of the pitch curve of each noncircular gear pair is greatly important, which directly affects the transmission ratio. Typical pitch curves of noncircular gear pairs include elliptic gear, eccentric gear, etc. And the pitch curves of the noncircular gear pair are determined according to the transmission ratios in this paper. If the transmission ratios are too complex, it will cause the values of m and n in Eq. (3) to be difficult, or even have no solution, and Eq. (3) is used to calculate the total transmission ratio. Consequently, the linear function, that is the first-order function, is chosen as the mathematical formula for the transmission ratio of noncircular gear pair. And a noncircular gear pair with the transmission ratio of the first functional form is named linear functional noncircular gear transmission.

Further, in the transmission system, gear 1 and gear 2 are fixed on the input shaft, and transmission ratios m_{ij} (i=1, 2; j=3, 4) of noncircular gear pairs 1–3 and 2–4 in working section $[0, t]$ are

$$m_{31} = \frac{\omega_3}{\omega_1} = f(\theta_1) = k_1\theta_1 + b_1, \quad \theta_1 \in [0, t], \tag{1}$$

$$m_{42} = \frac{\omega_4}{\omega_2} = g(\theta_1) = k_2(\theta_1 - \theta) + b_2, \quad \theta_1 \in [0, t], \tag{2}$$

where, θ_1 is the rotational angle of gear 1 and gear 2. θ is the phase angle between gear 1 and gear 2, and t is the maximum rotational angle at working section in a period; k and b represent the slope value and the intercept value of linear function, respectively.

By designing the double-row 2K-H type epicyclical gear train with a specific number of teeth, the following relation, which is the total transmission ratio, can be achieved:

$$m_{61} = m(k_1\theta_1 + b_1) + n(k_2\theta_2 + b_2) = c. \tag{3}$$

When a fixed rotational speed is inputted by shaft 1, shaft 6 will output a constant rotational speed, and the rotational speeds of each level in the gear transmission system are shown in Figure 2.

As shown in Figure 2, the continuously variable velocity outputting can be realized at phase angle θ using the second-order noncircular gear pair. And then the constant rotational speed can be outputted by output shaft 6, as the phase angle between two noncircular gear pairs is θ. And the wider range of constant rotational speed outputting can be achieved during each period with decreased phase angle.

2.2 Transmission Ratio of Noncircular Gear in Non-working Section

Obviously, the two pitch curve sections of the noncircular gear are different during a period, that is, the transmission ratio in the working section is a linear function, $\theta_1 \in [0, t]$, or that presents a complex curve in non-working section, $\theta_1 \in (t, T]$. Empirically, polynomial has been used to construct the transmission ratio function in non-working section outside the given interval in a transmission ratio period. Supposing the transmission ratio of noncircular gear pair in one period is expressed as follows.

$$m_{31} = \frac{n_3}{n_1} = \begin{cases} m_1(\theta_1) = k_1\theta_1 + b_1, & \theta_1 \in [0, t], \\ m_2(\theta_1) = \sum\limits_{i=0}^{6} \left(a_i\theta_1^i\right), & \theta_1 \in (t, T], \end{cases} \tag{4}$$

where, a_i is the polynomial coefficient.

The first-order and second-order noncircular gears are driving and driven parts, respectively. It is reported by Huang et al. [25] that the values of $a_i (i = 1, ..., 7)$ can be calculated by 7 independent equations including 1 closed condition and 6 boundary conditions, as shown in Eq. (5).

$$\begin{cases} \int_t^T m_2(\theta_1)\,\mathrm{d}\theta_1 = \frac{2\pi}{n_2} - \int_0^t m_1(\theta_1)\,\mathrm{d}\theta_1\,, \\ m_1(t) = m_2(t), \\ m_1'(t) = m_2'(t), \\ m_1''(t) = m_2''(t), \\ m_1(0) = m_2(T), \\ m_1'(0) = m_2'(T), \\ m_1''(0) = m_2''(T), \end{cases} \tag{5}$$

where, n_2 represents the order of noncircular gear. And so far, the transmission ratio of noncircular gear pair in the first period has been determined.

2.3 Influence Parameters of Pitch Curve of Noncircular Gear Pair

Due to the pitch curves form of two noncircular gear pairs is the same as Eq. (4) but some parameters are discrepant. Then, the transmission characteristics of noncircular gear pairs with linear functional transmission ratio are discussed based on the example of gear pairs 1–3. And the pitch curve of the noncircular gear pair can be obtained according to the centrode. Furthermore, in the working section $\theta_1 \in [0, t]$, the radius vectors of the pitch curves, can be expressed as:

$$r_1(\theta_1) = E\frac{m_{21}}{m_{21} + 1} = E\frac{k_1\theta_1 + b_1}{k_1\theta_1 + b_1 + 1}, \tag{6}$$

$$\begin{cases} r_2(\theta_1) = E - r_1(\theta_1) = \frac{E}{k_1\theta_1 + b_1 + 1}, \\ \theta_2(\theta_1) = \int_0^{\theta_1} (k_1\theta + b_1)\mathrm{d}\theta = \frac{k_1}{2}\theta_1^2 + b_1\theta_1, \end{cases} \tag{7}$$

where,$r_1(\theta_1)$ is the radius vector of driving gear, which numbered 1; $r_2(\theta_1)$ is the radius vector of driven gear, which numbered 2, and $\theta_2(\theta_1)$ is the rotational angle of driven gear; the center distance between two gears is E. According to Eq. (5), k_1 and b_1 will be two important parameters of transmission ratio in non-working section. By analyzing the influence of the parameters k_1 and b_1 on the pitch curve in non-working section of noncircular gear pair with linear functional transmission ratio, the optimal pitch curve can be acquired further.

(1) The influences of parameter b_1 on transmission ratio of noncircular gear pair, when k_1 is constant.

Taking 1–3 gear pairs as an example, the value of period T is π, and the maximum rotational angle t in working section during a period is 2/3π. When k_1 is 3/4π , the influence of parameter b_1 with variable value on transmission ratio is analyzed.

As shown in Figure 3, with the value of b_1 increases in the interval of [0.5, 2.1], the amplitude of transmission ratio first decreased and then increased, and the transmission ratio curve in non-working section is first convex and then concave. Nevertheless, it seems clear that the transmission ratio curve changes gently with the rotation angle around the value of 1.75 for b_1. And the transmission in this situation will be more suitable, because the instantaneous velocity and acceleration are smaller, and the minimum radius of curvature of pitch curve is larger. The corresponding transmission ratio and pitch curve of noncircular gear pair are shown as red curve in Figure 3(a) and (b), respectively.

The interval length y of codomain of transmission ratio can be calculated as Eq. (8).

$$y = m_{31\max} - m_{31\min}. \tag{8}$$

According to the above results, when the transmission ratio with the most appropriate linear function parameters is obtained, the interval length of the codomain of the transmission ratio is the narrowest. Therefore, the minimum interval length of the codomain of the transmission ratio under different parameters can help us to get the value of b_1 that best fits for transmission, at a certain value of k_1.

(2) *The influences of k_1 on transmission ratio of noncircular gear pair, when b_1 is constant.*

Similarly, when b_1 is 0.5, with different values of k_1, the influences of k_1 on transmission ratio are analyzed as shown in Figure 4.

With the increase of k_1 from $3/4\pi$ to $19/4\pi$, the amplitude of the transmission ratio first decreases and then increases. The transmission ratio curve in the non-working section is first convex and then concave. And around the value of $9/2\pi$ for k_1, the transmission ratio curve is the gentle, that is, the interval length y of codomain of transmission ratio is narrow, which is suitable for transmission. Then, while k_1 is $9/2\pi$ and b_1 is 0.5, the transmission ratio function curve is relatively smooth, and the transmission ratio and centrode of noncircular gear pair are shown as blue curve in Figure 4.

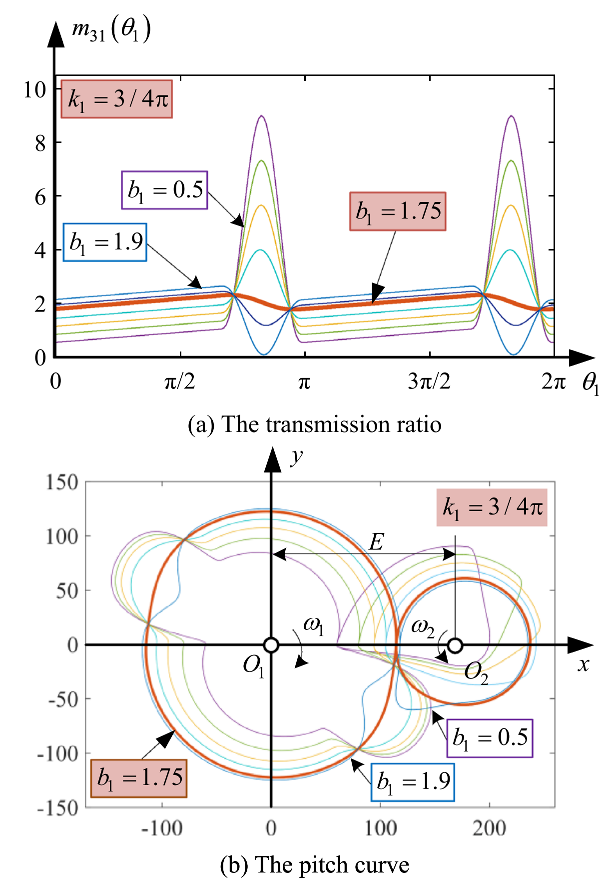


Figure 3 The influence of b_1 on transmission ratio of noncircular gear pair

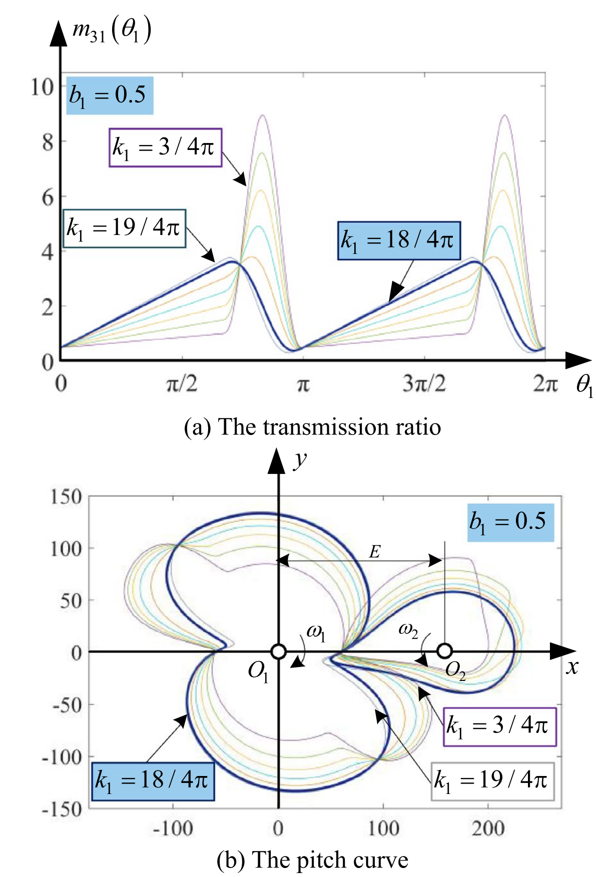
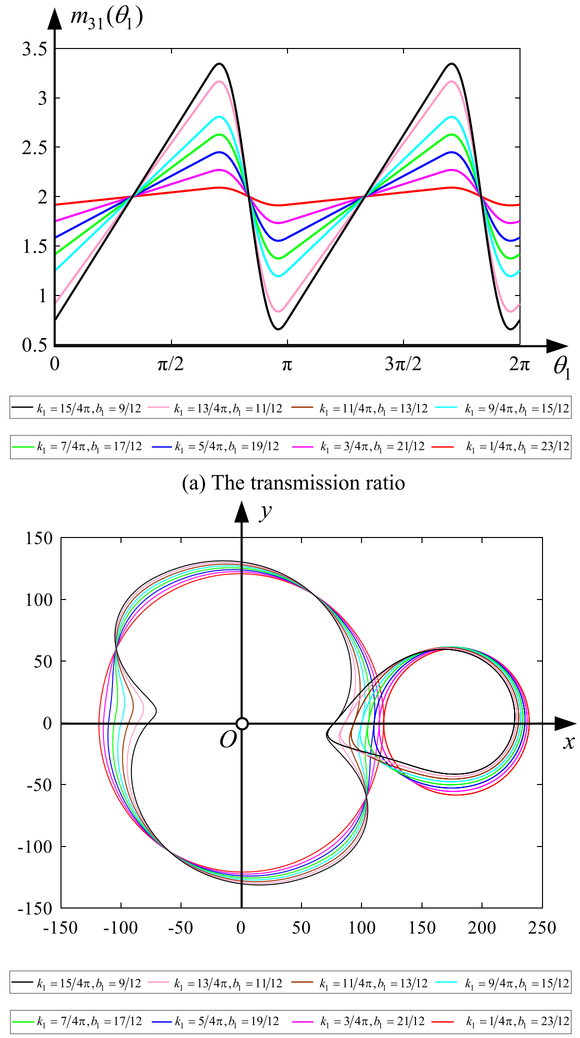


Figure 4 The influences of k_1 on transmission ratio of noncircular gear pair

(3) *The combined influences of* k_1 *and* b_1 *on transmission ratio of noncircular gear pair*

According to the above analysis, each value of k_1 must correspond to an optimal value of b_1 for transmission, so a series of values of k_1 and corresponding b_1 can be obtained. When the maximum rotational angle Φ_2 in working section of the driven gear is equal to the product of the maximum rotational angle Φ_1 of the driving gear and its order n_1, that is $\Phi_2 = \Phi_1 \times n_1$, the optimal transmission ratio curve of the non-working section can be acquired. Furthermore, when the transmission ratio of the gear pair in the working section is expressed as $m_{21} = k_1\varphi_1 + b_1$, and the transmission ratio curve of non-working section constructed by the six-order polynomial meets the condition that $k_1\pi + 3b_1 = 6$, the optimal transmission ratio curve in non-working section will also be obtained. At this point, the range of transmission ratio of noncircular gear pair in the working section is $[b, 4 - b]$; in addition, with a certain value of k_1, b_1 is $2 - (\pi/3)k_1$; and the range of transmission ratio is $[2 - (\pi/3)k_1, 2 + (\pi/3)k_1]$.

The comparisons of pitch curves of noncircular gear pairs with different transmission ratio parameters are shown in Figure 5. With k_1 gradually increasing, even if the most appropriate b_1 is taken to make the minimum fluctuation of the transmission ratio curve, and the transmission ratio function at this point is smooth, the minimum curvature radius of the pitch curve is so small that the teeth are subjected to extra bending stress, resulting that the teeth are prone to bending and fracture, which is not suitable for transmission.



(a) The transmission ratio

(b) The pitch curve

Figure 5 Comparisons of transmission ratio and centrode of noncircular gear pairs with different transmission ratio parameters

3 The Contact Ratio of Noncircular Gear Pair

The contact ratio of one noncircular gear pair is the length ratio between the effective meshing curve and the pitch of the gear base circle, which significantly affects the meshing performance of noncircular gear transmission. It was reported by Wu et al. [9], the contact ratio of a noncircular gear pair can be expressed as follows:

$$\varepsilon = \frac{u_1 + u_2}{\pi m \cos\alpha_0}, \tag{9}$$

where, $u_i = \sqrt{(\rho_i + h_{ai})^2 - (\rho_i \cos\alpha_0)^2} - \rho_i \sin\alpha_0, (i = 1, 2)$; ρ_1 and ρ_2 are the curvature radius of the pitch curves at the meshing point P; α_0 is the tangent angle between meshing curve and the pitch curve at meshing point P, that is, the tooth profile angle of rack cutter.

According to Eq. (9), some main parameters that affect the contact ratio of involute noncircular gear transmission are center distance E, transmission ratio coefficient k_1 and b_1, addendum coefficient h_a^*, modification coefficient x, module m, and tooth profile angle of the rack cutter α_0. Considering that the center distance E and the expected transmission ratio function m_{21} are unchanged, the pitch curve of modified gear pair should be same as that of the original gear pair. To satisfy the above-mentioned conditions, when one noncircular gear with positive addendum modification has been designed, the other one must be processed by negative addendum modification. Furthermore, the absolute value of the modification coefficient of the two gears should be equal. So only the height modification method with the modification coefficient $x_2 = -x_1$ and total modification coefficient $\Sigma x = x_1 + x_2 = 0$ can be used [9]. As shown in Figure 6, zero modification has little effect on the contact ratio of noncircular gear.

Therefore, for a noncircular gear pair with a determined pitch curve, the main parameters affected by the contact ratio are as follows: tooth profile angle of rack cutter α_0, modification coefficient h_a^* and module m. When parameters of noncircular gear pair according to the optimality principle are selected as: $k_1 = 3/4\pi$, $b_1 = 1.75$, $E = 180$ mm, the influence of other parameters on the contact ratio of linear functional noncircular gear pair is shown in Figure 7.

As shown in Figure 7, for a certain noncircular pitch curve, with the increase of the addendum coefficient h_a^* or the number of teeth z, the decrease of tooth profile angle α_0 of the rack cutter or the modulus m, the minimum contact ratio of the noncircular gear pair in the working section will increase. In this paper, it is defined as *high contact ratio* while the minimum instantaneous contact ratio is greater than 2. When parameters are taken as the surface above plane $\varepsilon = 2.0$ shown in Figure 7(b), gear transmission with high contact ratio can be achieved.

With respect to different linear functional pitch curves, the influence of the transmission ratio coefficient on contact ratio is shown in Figure 8. It is obvious that the curvature variations in the working section increase with the transmission ratio coefficient k_1 increasing, and the smaller minimum contact ratio is also acquired.

4 Involute Profile of Noncircular Gear

Based on the pitch curve of the noncircular gear, the involute tooth profile of the noncircular gear can be obtained. The design of a noncircular gear tooth profile is to determine its geometric parameters and make it have reasonable meshing performance. For example, the tooth profile of the noncircular gear is required to ensure that the two calculated curves are pure rolling (i.e. transmission according to the required transmission ratio) with an appropriate pressure angle. Indeed, the noncircular gear pair transmission

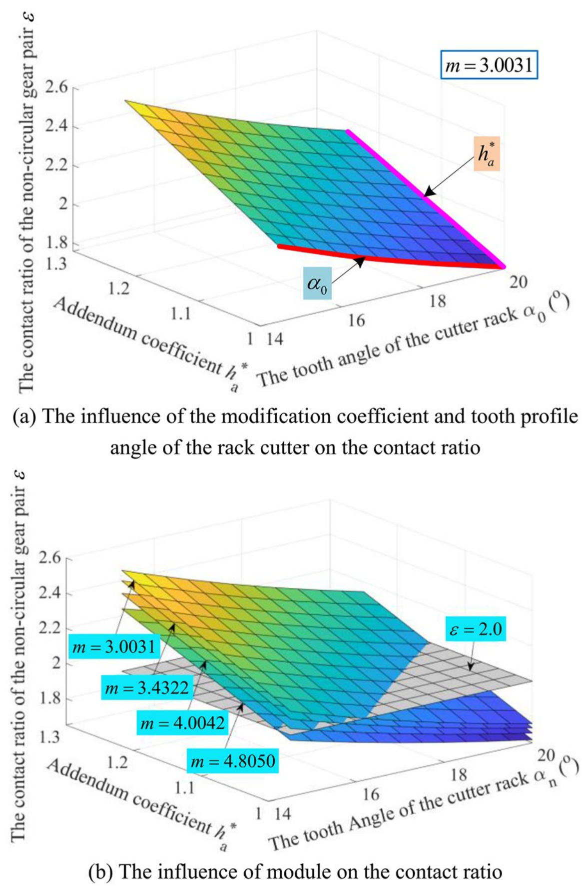


Figure 7 The influence of each parameter on the contact ratio

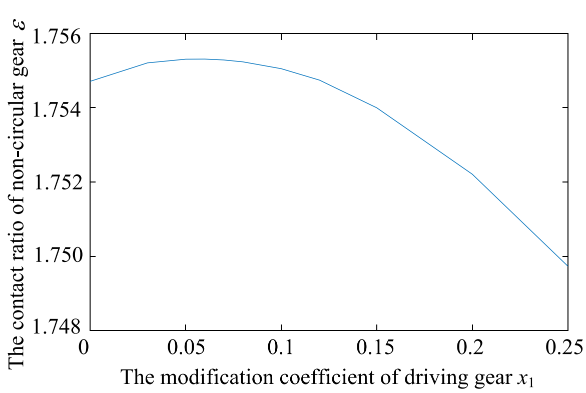


Figure 6 The influence of zero modification on the contact ratio of noncircular gear

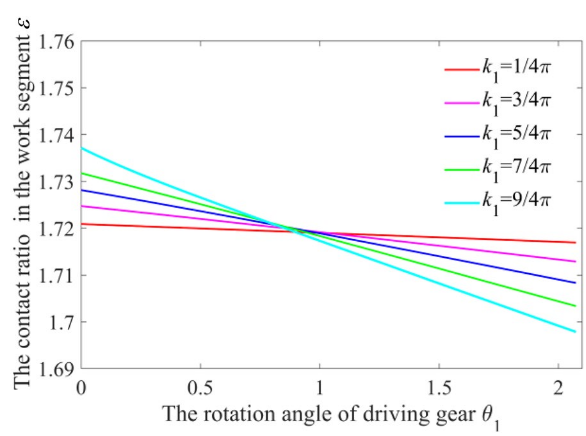


Figure 8 The influence of transmission ratio coefficient on contact ratio

is directionality when it has meshed correctly. While the driving gear rotates clockwise, the transmission ratio of the gear pair is a required linear function. At this time, both the driving gear and the driven gear mesh with the right tooth profile. Therefore, the right tooth profile of the driving gear will be taken as the research object in this paper, and for simplicity, the following tooth profiles without special descriptions are called the right tooth profile of a driving gear.

4.1 Involute Profile of Driving Gear

In noncircular gear pair transmission, coordinate system $S_1(O_1 - x_1y_1z_1)$ is rigidly connected to the driving gear, and its initial position is shown in Figure 9. In the initial position, polar axis of the driving gear coincides with the y-axis of the fixed coordinate frame $S_0(O - xyz)$, and the x_1 -axis of coordinate system S_1 is parallel to the x-axis of the fixed coordinate frame with a spacing of $r_1(0)$. To simplify calculation, the right profile of the 1# tooth passes through the instantaneous center O at the initial position, then O must be a point on the meshing curve.

As shown in Figure 9, one right tooth profile on the driving gear intersects with pitch curve at point P_0. M is the point on this tooth profile outside the pitch curve, and M' is the point on this tooth profile within the pitch curve. Taking the point M for example, the normal n_Mn_M of point M intersects the pitch curve at point P, which is the instantaneous center of the gear pair when point M is the meshing point. Moreover, the tangent of point P on the pitch curve is named $\tau_P\tau_P$. In the coordinate system S_1, the equation of the tooth profile $\boldsymbol{r}_{1R}$ is

$$\boldsymbol{r}_{1R} = \boldsymbol{O_1M} = \boldsymbol{O_1P} + \boldsymbol{PM}, \tag{10}$$

where, $\boldsymbol{O_1P} = \boldsymbol{r}_1(\phi_1)$, and in terms of involute property,

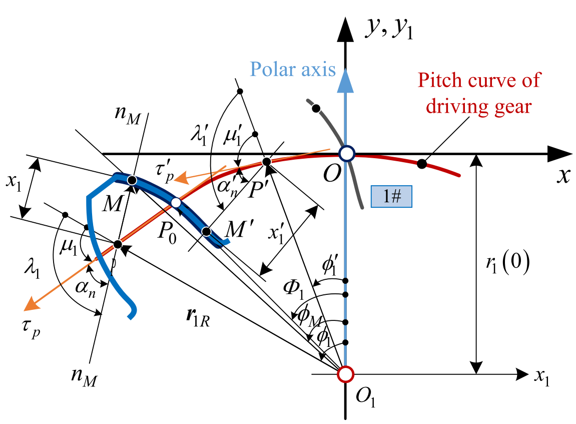


Figure 9 The analytical diagram of tooth profile of driving gear

$$x_1 = |\boldsymbol{PM}| = \cos\alpha_n \cdot \int_{\Phi_1}^{\phi_1} \sqrt{{r_1'}^2(\phi) + r_1^2(\phi)}\mathrm{d}\phi. \tag{11}$$

It is noteworthy that $x_1' = |\boldsymbol{P'M'}| = \widehat{P'P_0}\cos\alpha_n = \cos\alpha_n \bullet \int_{\phi_1'}^{\Phi_1} \sqrt{r_1'^2(\phi) + r_1^2(\phi)}\mathrm{d}\phi$ while the point M' on this tooth profile is within the pitch curve owing to $\phi_1' < \Phi_1$.

In addition, at the meshing point P, the angle λ_1 between the radial vector $\boldsymbol{O_1P}(=\boldsymbol{r}_1(\phi_1))$ of the pitch curve and the normal n_Mn_M of the right tooth profile is

$$\lambda_1 = \alpha_n + \mu_1, \tag{12}$$

where, μ_1 is the angle between the radial vector $\boldsymbol{r}_1(\phi_1)$ of the pitch curve and the positive direction of tangent vector $\boldsymbol{\tau}_p$ at point P, and the measuring direction of angle λ_1 is same as that of angle μ_1; the angle α_n represents the tooth profile angle of cutter.

Further, the radial vector of the tooth profile $\boldsymbol{r}_{1R}$ in the coordinate system S_1 can be obtained as

$$\begin{aligned} \boldsymbol{r}_{1R}^{(1)} &= \begin{bmatrix} x_{1R} \\ y_{1R} \end{bmatrix} = \begin{bmatrix} |\boldsymbol{r}_{1R}|\sin\phi_M \\ |\boldsymbol{r}_{1R}|\cos\phi_M \end{bmatrix} \\ &= \begin{bmatrix} -r_1(\phi_1)\sin\phi_1 \pm x_1\sin(\phi_1 + \lambda_1) \\ r_1(\phi_1)\cos\phi_1 \mp x_1\cos(\phi_1 + \lambda_1) \end{bmatrix}, \end{aligned} \tag{13}$$

where the above symbol is used for the point outside the pitch curve; and the below symbol is used for the point within the pitch curve.

Consequently, the polar coordinate equation of the right tooth profile of the driving gear is as follows:

(1) While the point M on the tooth profile is outside the pitch curve

$$\begin{cases} r_{1R}(\phi_1) = \sqrt{r_1^2(\phi_1) + x_1^2 + 2r_1(\phi_1)x_1\cos\lambda_1}, \\ \phi_M(\phi_1) = \arctan\dfrac{-r_1(\phi_1)\sin\phi_1 + x_1\sin(\phi_1 + \lambda_1)}{r_1(\phi_1)\cos\phi_1 - x_1\cos(\phi_1 + \lambda_1)}. \end{cases} \tag{14}$$

(2) While the point M' on the tooth profile is within the pitch curve

$$\begin{cases} r_{1R}(\phi_1) = \sqrt{r_1^2(\phi_1) + x_1^2 - 2r_1(\phi_1)x_1\cos\lambda_1}, \\ \phi_M(\phi_1) = \arctan\dfrac{-r_1(\phi_1)\sin\phi_1 - x_1\sin(\phi_1 + \lambda_1)}{r_1(\phi_1)\cos\phi_1 + x_1\cos(\phi_1 + \lambda_1)}. \end{cases} \tag{15}$$

In the same way, the polar coordinate equation of the right tooth profile of the driven gear also can be obtained.

(1) While the point M on the tooth profile is outside the pitch curve

$$\begin{cases} r_{2R}(\phi_1) = \sqrt{r_2^2(\phi_1) + x_2^2 + 2r_2(\phi_1)x_2\cos\lambda_2}, \\ \phi_{2M}(\phi_1) = \arctan\dfrac{-r_1(\phi_1)\sin\phi_2 - x_2\sin(\phi_2+\lambda_2)}{r_2(\phi_1)\cos\phi_2 + x_2\cos(\phi_2+\lambda_2)}. \end{cases} \tag{16}$$

(2) While the point M' on the tooth profile is within the pitch curve

$$\begin{cases} r_{2R}(\phi_1) = \sqrt{r_2^2(\phi_1) + x_2^2 - 2r_2(\phi_1)x_2\cos\lambda_2}, \\ \phi_{2M}(\phi_1) = \arctan\dfrac{-r_1(\phi_1)\sin\phi_2 + x_2\sin(\phi_2+\lambda_2)}{r_2(\phi_1)\cos\phi_2 - x_2\cos(\phi_2+\lambda_2)}. \end{cases} \tag{17}$$

4.2 The Meshing Curve of Noncircular Gear Pair

As is known, each pair of conjugate tooth profiles of noncircular gear pair is generally different, so its meshing curve is also different [8].

As shown in Figure 10, at the initial position of noncircular gear pair transmission, one point M'' on the right tooth profile of the driving gear whose polar coordinates is (ϕ_M, r_{1R}), and the normal of the right tooth profile and pitch curve intersect at P''. When the driving wheel has rotated angle θ_1 clockwise, which θ_1 is equal to ϕ_1 with opposite direction, P'' turned to P, M'' turned to M, and the radial vector of pitch curve coincides with the y-axis of fixed coordinate frame S_0 at this moment. According to the principle of gearing, when point P falls on the line between the centers of the two gears, point M on the tooth profile of the driving gear is tangent to the corresponding point on its conjugate tooth profile, and point M is one point on the meshing curve of this noncircular gear pair.

The rotating angle of driving gear θ_1 is the angle between reference frame S_0 and S_1. The coordinate transformation matrix $\boldsymbol{M}_{01}$ from S_1 to S_0 is as follows:

$$\boldsymbol{M}_{01} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & r_1(0) \\ 0 & 0 & 1 \end{bmatrix}. \tag{18}$$

By transforming the radial vector of the tooth profile $\boldsymbol{r}_{1R}$ into the fixed coordinate frame S_0, the meshing curve of the gear pair can be obtained by

$$\boldsymbol{r}_M = \boldsymbol{M}_{01}\boldsymbol{r}_{1R}^{(1)} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & r_1(0) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1R} \\ y_{1R} \\ 1 \end{bmatrix}. \tag{19}$$

By the way, because θ_1 is equal to ϕ_1, the meshing curve of the noncircular gear pair in fixed coordinate frame S_0 can be expressed as

$$\begin{cases} x_M = x_{1R}\cos\phi_1 + y_{1R}\sin\phi_1, \\ y_M = -x_{1R}\sin\phi_1 + y_{1R}\cos\phi_1 + r_1(0). \end{cases} \tag{20}$$

While the number of teeth z_1 and z_2 are 46 and 23 respectively, the theoretical results of some meshing curves have been obtained by MATLAB, as shown in Figure 11. The azure curve is the meshing curve of 1# tooth, and above are the meshing curves from 2# tooth to 18# tooth, where, the teeth from 2# to 16# are fully on the working section, both the upper half of 1# tooth profile and lower half of 17# tooth profile are in the working section, and the 18# tooth is completely on the non-working section as the red curve in Figure 11.

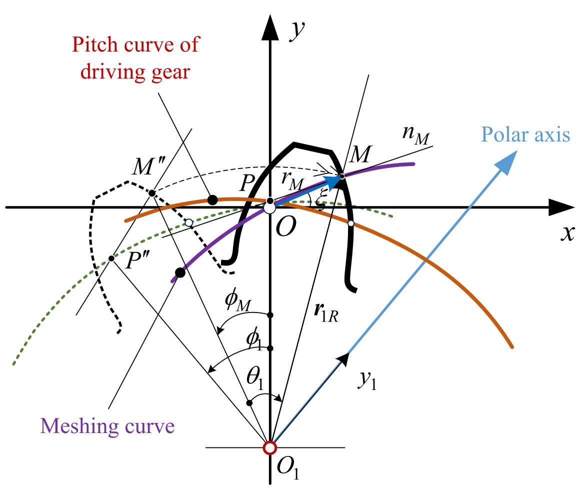


Figure 10 The analytical diagram of meshing curve of noncircular gear pair

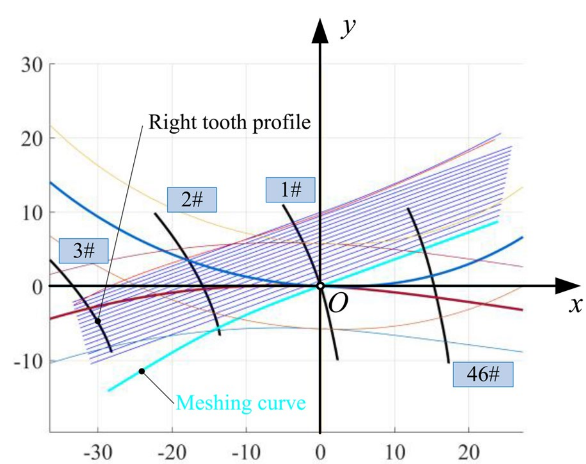


Figure 11 The theoretical results of some meshing curves of noncircular gear pair by MATLAB

4.3 The Length of Contact Profile of Driving Gear

The meshing process of one conjugate tooth profiles pair is from an addendum of driving gear peak contacting to an addendum of driving gear peak contacting. Therefore, in the fixed coordinate frame S_0, the intersection point F of the meshing curve and the addendum curve of the driven gear is the starting point of the contact profile of the driving gear, and the intersection point A of the addendum curve and the tooth profile of the driving gear is the end point of the contact profile of the driving gear. 2# tooth is taken as an example as shown in Figure 12.

The addendum curve [9] of the driven gear in the reference system $S_2(O_2 - x_2y_2z_2)$, which is rigidly attached to the driven gear, can be presented as

$$\begin{cases} r_{2A}(\varphi_2) = \sqrt{r_2^2(\varphi_2) + h_a^2 + 2r_2(\varphi_2)h_a \sin\mu_2}, \\ \varphi_{2A}(\varphi_2) = \varphi_2 - \arcsin\dfrac{h_a\cos\mu_2}{r_{2A}(\varphi_2)}. \end{cases} \tag{21}$$

The following simultaneous equations can be used to get the intersection point of the tooth profile as Eq. (16) and the addendum curve as Eq. (21) of the driven gear.

$$\begin{cases} r_{2R}(\phi_1) = r_{2A}(\varphi_2), \\ \phi_{2M}(\phi_1) = \varphi_{2A}(\varphi_2). \end{cases} \tag{22}$$

Then a unique solution (ϕ_1, φ_2) can be obtained from the above equations. Substituting the solution into the tooth profile equation of the driven gear, the tooth addendum point $F(\phi_{2F}(\phi_1), r_{2F}(\phi_1))$ of the driven gear can be obtained. According to the principle of gears, the solution of ϕ_1 is the pitch curve angle ϕ_{1F} when the tooth of the driving gear enters into meshing.

Similarly, the tooth addendum point $A(\phi_{1A}(\phi_1), r_{1A}(\phi_1))$ of the driving gear also can be obtained, and at this time, the value of ϕ_1 is the pitch

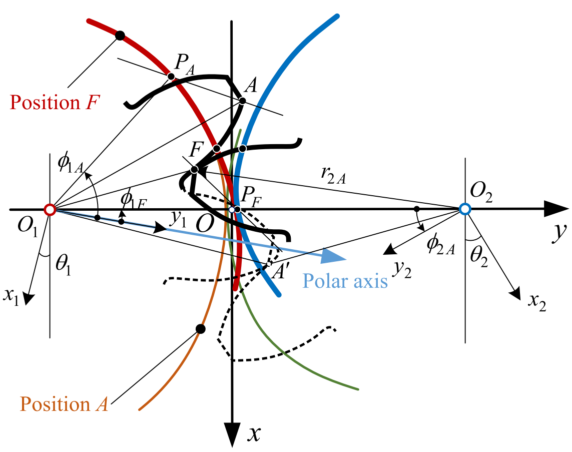


Figure 12 The analytical diagram of contact profile of driving gear and the addendum curve of driven gear

curve angle ϕ_{1A} when the tooth of driving gear is out of meshing. And the theoretical length of the contact profile of the tooth l_1 can be obtained further as follows.

$$l_1 = \int_{\phi_{1F}}^{\phi_{1A}} \sqrt{x'^2_{1R} + y'^2_{1R}}\mathrm{d}\phi_1. \tag{23}$$

As shown in Figure 13, the 2# theoretical contact tooth profile $\overset{\frown}{P'P_0}$ of the driving gear has been obtained by MATLAB. In addition, according to the results of contact stress by ANSYS, there are double tooth-meshing areas and three tooth-meshing areas on the actual meshing curve $\overset{\frown}{P_1P_2'}$.

5 Experimental Verification

Combined with the above analysis, in order to achieve high contact ratio transmission, the noncircular gear pair parameters are selected as shown in Table 1.

The tooth profile curve of the noncircular gear pair obtained by simulating the machining process of the

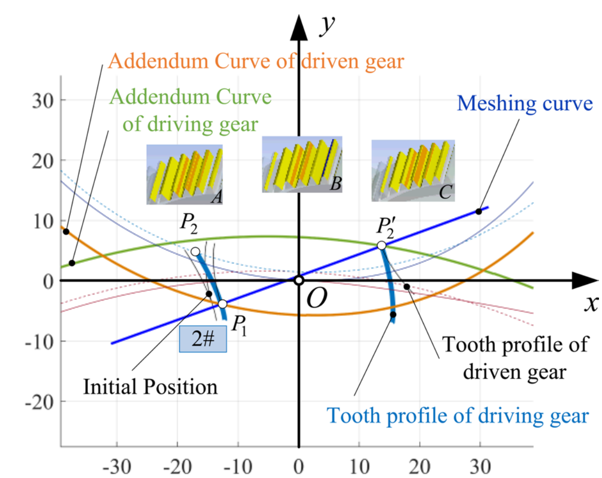


Figure 13 The 2# theoretical contact tooth profile of driving gear by MATLAB

Table 1 Parameter values of noncircular gear pair

Parameter	Value
k_1	3/4π
b_1	1.75
E(mm)	172.32
m	5
z_1	46
z_2	23
α	18°
h_a^*	1.15

slotting cutter with MATLAB is shown in Figure 14. The tooth profile points of the noncircular gear pair obtained by MATLAB are imported into SolidWorks to establish the model, and the final model of the noncircular gear pair used for machining will be obtained after smoothing the tooth surface.

To measure the transmission ratio and the contact length of the tooth profile of the noncircular gear pair, after finishing the processing of the noncircular gear pair through the five-axis CNC machining center, the noncircular gear rolling test was arranged on the comprehensive experimental test platform of the noncircular gear transmission, and the experimental test platform is shown in Figure 15. By the way, to print the top line mark of the driving gear tooth more clearly on the driven gear tooth surface, the rotating speed of the input gear in this experiment is 60 r/min, which is slower than normal operation.

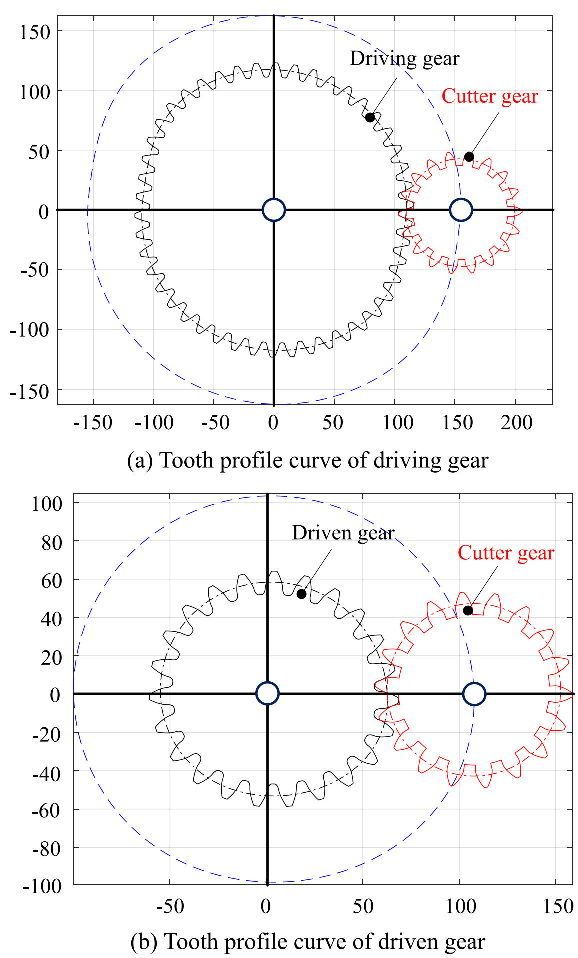


Figure 14 The tooth profile curve of noncircular gear pair with MATLAB.Tooth profile curve of driven gear

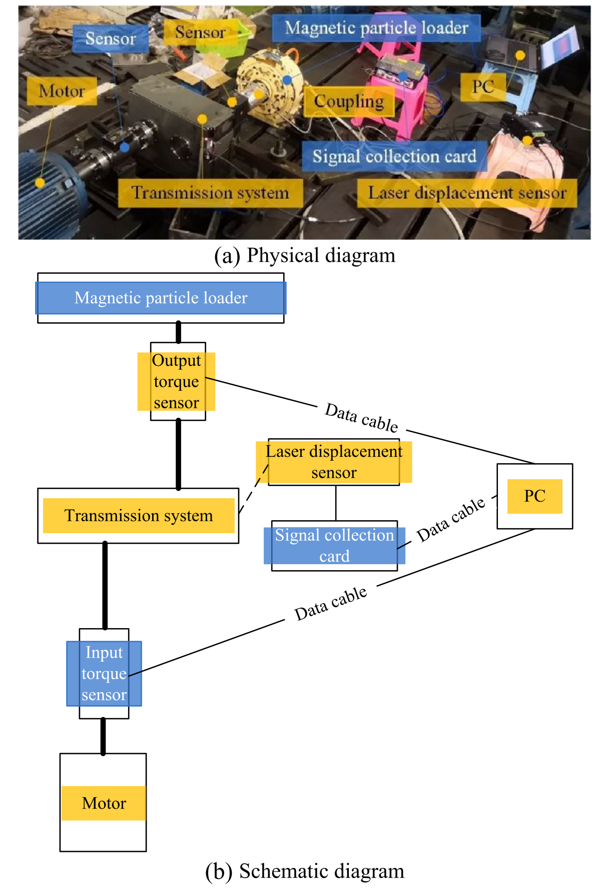


Figure 15 The experimental test platform of the noncircular gear transmission

During this experiment, the input torque and speed provided by the drive motor drive the driving gear to rotate firstly, through gear meshing, the driven gear drives the magnetic particle loader to rotate subsequently. The input torque tachometer installed between the drive motor and gear case can measure the input torque, rotational velocity, and other parameters. The output torque tachometer installed between the magnetic particle loader and gear case can measure the output torque, rotational velocity, and other parameters. Finally, the input/output torque and rotational velocity are fed back to the operator console for post-processing. And these parameters including load torque and speed of the drive motor can be further adjusted by the operation console.

To minimize the influence of error caused by random signal interference on experimental results, the acquired experimental data must be processed. Therefore, the actual transmission ratio i'_{12} is obtained from Eq. (24).

$$i'_{12} = \frac{\omega'_1}{\omega'_2} = \frac{n'_1}{n'_2}, \tag{24}$$

where, ω'_1 and n'_1 are angular velocity and rotating speed of driving gear, respectively. ω'_2 and n'_2 are angular velocity and rotating speed of driven gear, respectively.

Due to the involute profile of the noncircular gear, the contact ratio error can be reflected by the contact profile length error. As is known, the actual meshing curve of the noncircular gear pair is very important for reflecting the contact ratio of this gear pair. According to the actual meshing curve of the gear pair, the partial tooth profile that participates in meshing can be known, that is, the working section of the tooth profile. Therefore, it is proposed to confirm the error of contact ratio indirectly by measuring the error of the working section of the tooth profile. The partial tooth surface containing the tooth addendum of the driven gear was coated with a uniform layer of red lead powder, which is shown in Figure 16(a).

After the noncircular gear pair turned a cycle, the contact marks with red lead powder were left on the teeth surface of the driving gear, that is, the initial position of the contact tooth profile, which is shown in Figure 16(b). Then the contact mark on the driving gear surface was recorded by A4 paper; after being scanned into the computer, the error of the contact tooth profile length was obtained through data processing subsequently. 3# teeth of noncircular gear pair were taken as examples, and the contact area and tooth surface were shown in Figure 16(c), where l_2 is the length of the experimental contact tooth profile, and l_0 is the length of the tooth profile.

Further, the length error of contact tooth profile e_ε could be obtained by analyzing the length of experimental contact profile l_2 and theoretical contact profile l_1 according to Eq. (25).

$$e_\varepsilon = \frac{|\Delta l|}{l_1} \times 100\% = \frac{|l_1 - l_2|}{l_1} \times 100\%. \tag{25}$$

Next, the experimental results after being processed are quantitatively analyzed. Figure 17 shows the comparisons between the theoretical and experimental results of the transmission ratio and contact tooth profile length of the noncircular gear pair.

The results in Figure 17 show that: (1) Except for the errors caused by measurement and installation, the experimental value of the transmission ratio of the noncircular gear pair is basically consistent with the theoretical value, and the period of the experimental value is consistent with its theoretical value. Because the experiment was carried out under the condition of low speed and light load, the influence caused by the dynamic characteristics of the noncircular gear can be ignored, and the error between the experimental results and the theoretical results is relatively small. In the working section, the maximum error of the transmission ratio is 6.18%, which in the non-working section is 5.19%, and the maximum error mainly lies in the connection between the working section and the non-working section.

(2) It can be seen from the comparison results of the contact tooth profiles of noncircular gears that in a meshing period, the experimental values of the contact tooth profiles of noncircular gear are basically consistent with the theoretical values, which verifies the feasibility of the theoretical analysis method. The fluctuation amplitude of the experimental contact tooth profiles is greater than the theoretical results, and the experimental contact tooth profiles are lower than the theoretical values, while

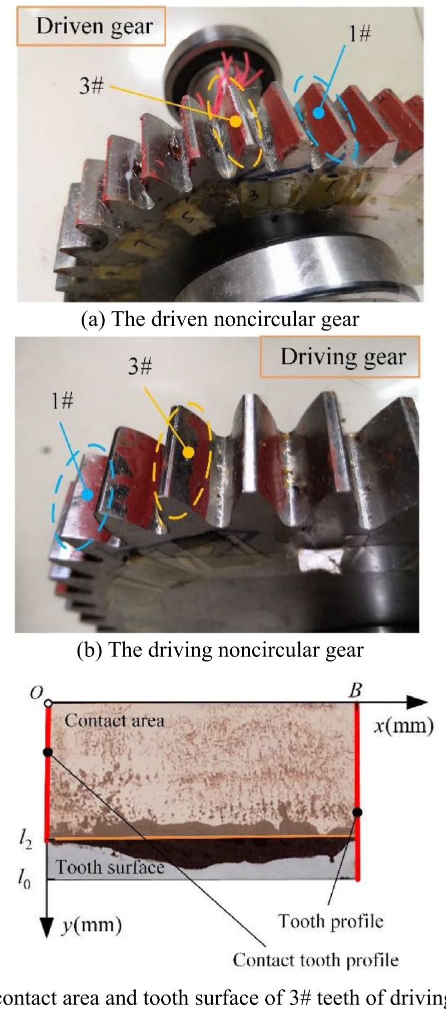


Figure 16 The length of contact profile of driving gear

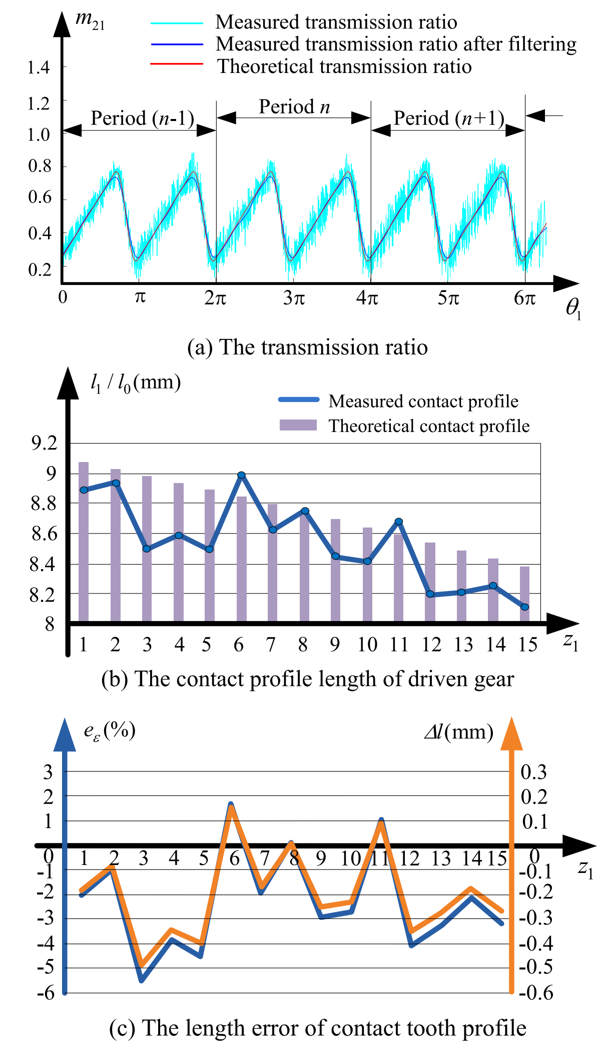


Figure 17 The transmission ratio and contact profile length of the driving gear

the maximum error in the working section is 5.4%. The main reason for this result is that in order to ensure the noncircular gear pair normal meshing, the proper backlash is required by slightly increasing the installment center distance, which makes the actual contact profile smaller. In addition, the gear machining accuracy is also an important factor leading to the actual contact profile length decreasing.

6 Conclusions

(1) The linear functional noncircular gear pair is used to realize continuously variable transmission by means of branch-differential coupling, and the pitch curves of this gear pair are designed according to the required transmission ratio. The influence of the parameters of the linear function, k_1 and b_1, on the pitch curve of the noncircular gear pair with linear functional transmission ratio in the non-working section is analyzed; further, the most suitable pitch curve form is obtained, which each value of k_1 must correspond to the best value of b_1.

(2) When the pitch curve in the non-working section constructed by a six-order polynomial meets the optimal conditions, the pitch curve is the smoothest. When k_1 increases gradually, even if b_1 is the most suitable value to make the variation trend of the pitch curve in the non-working section has the gentlest change trend, however, the minimum curvature radius of the pitch curve of driving/driven gear is too small. Thus, on the one hand, the strength of gear teeth is insufficient; on the other hand, the instantaneous speed and acceleration are too large, resulting in unstable transmission. Generally, this method can be used to design the noncircular gear pair which is suitable for CVT.

(3) The larger the transmission ratio parameter k_1 of the linear functional noncircular gear pair is, the larger the variation range in the working section is, and the smaller the minimum contact ratio is. Finally, by measuring the length error of contact tooth profile, the error of the contact ratio is detected indirectly, which could verify the correctness of the design of a high contact ratio of noncircular gear pair.

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Author Contributions
CL and YH were in charge of the whole trial; YH and CH wrote the manuscript; YY and ZC assisted with sampling and laboratory analyses. All authors read and approved the final manuscript.

Authors' Information
Yanan Hu, was born in 1993. She received her Ph.D. degree in mechanical engineering from *State Key Laboratory of Mechanical Transmission, Chongqing University, China*, in 2021. Now, she is a lecturer at *Binzhou University, China*. Her research interests include theoretical research and application design of noncircular gear and curve-face gear composite transmission.
Chao Lin, born in 1958, is a professor and doctoral supervisor at *College of Mechanical Engineering* / the *State Key Laboratory of Mechanical Transmission, Chongqing University, China*. His research interests include meshing theory of spiral bevel gears, noncircular gears and new gear transmission, and precision transmission and drive.
Chunjiang He, born in 1992, received the B.S. degree in mechanical engineering from *Chongqing University, China*, in 2011, and Ph.D. degree in mechanical engineering from *Chongqing University, China*, in 2020. Now, he is a lecturer at *School of Mechanical and Power Engineering, Chongqing University of Science and Technology, University*. His research interests include intelligent design and manufacturing.
Yongquan Yu, born in 1994, received the B.S. degree in mechanical engineering from *Hefei University of Technology, China*, in 2016. He is currently pursuing the Ph.D. degree in mechanical engineering at *Chongqing University, China*.

His research interests include the development of the new type of gear transmission, fundamental study of curve-face gear, and surface topography of noncircular gear.
Zhiqin Cai, born in 1988, is a lecturer at the *School of Aeronautics and Astronautics, Xiamen University, China*. He received his PhD in mechanical engineering from *Chongqing University, China*. His research interests include intelligent design of precision gear driven by shape coupling, micro-texture of tooth surface, and energy-saving transmission design.

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Declarations

Competing Interests
The authors declare no competing financial interests.

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References
[1] L Jian, K T Chau. Design and analysis of a magnetic-geared electronic-continuously variable transmission system using finite element method. *Progress in Electromagnetics Research*, 2010, 107: 47–61.
[2] Z Ruan. *Mechanical continuously variable transmission*. Beijing: China Machine Press, 1983. (in Chinese)
[3] T Fujii, T Kurokawa, S Kanehara. A study of a metal pushing V-belt type CVT–part 1: relation between transmitted torque and pulley thrust. *SAE Technical Papers*, 1993.
[4] A Yildiz, A Piccininni, F Bottiglione, et al. Modeling chain continuously variable transmission for direct implementation in transmission control. *Mechanism and Machine Theory*, 2016, 105: 428–440.
[5] X B Chen, P Hang, Wei Wang, et al. Design and analysis of a novel wheel type continuously variable transmission. *Mechanism and Machine Theory*, 2017, 107(4800): 13–26.
[6] H Machida, H Itoh, T Imanishi, et al. Design principle of high power traction drive CVT. *SAE Technical Papers*, 1995: 1365–1375.
[7] F L Litvin, I Gonzalez-Perez, Alfonso Fuentes, et al. Design and investigation of gear drives with non-circular gears applied for speed variation and generation of functions. *Computer Methods in Applied Mechanics and Engineering*, 2008, 197(45–48): 3783–3802.
[8] F S Li. *Design of noncircular gear and special gear transmission*. Beijing: China Machine Press, 1983. (in Chinese)
[9] X T Wu, G H Wang. *Noncircular gear and non-uniform velocity ratio transmission*. Beijing: China Machine Press, 1997. (in Chinese)
[10] F L Litvin, A Fuentes. *Gear geometry and applied theory*. Cambridge University Press, 2004.
[11] G X Sun, C Q Sun, B Li, et al. Design research of noncircular gears on continuously variable transmission. *Journal of Mechanical Transmission*, 2015, 9: 62–65. (in Chinese)
[12] T Emura, A Arakawa. A new steering mechanism using noncircular gears. *JSME International Journal Series Iii-Vibration Control Engineering Engineering for Industry*, 1992, 35,(4): 604–610.
[13] H Z Song, J G Zheng, Shi Wei, et al. Design and research of variable speed device based on spiral noncircular gear. *Journal of Mechanical Engineering*, 2017, 53(23): 101. (in Chinese)
[14] M Addomine, G Figliolini, E. Pennestrì, et al. A landmark in the history of non-circular gears design: the mechanical masterpiece of dondi's astrarium. *Mechanism and Machine Theory*, 2018, 122: 219–232.
[15] D B Dooner, H D Yoon, A. Seireg, et al. Kinematic considerations for reducing the circulating power effects in gear-type continuously variable transmissions. *Proceedings of the Institution of Mechanical Engineers, Part D: Journal of Automobile Engineering*, 1998, 212(6): 463–478.
[16] R Ferguson, L Daws, J H Kerr, et al. The design of a stepless transmission using non-circular gears. *Mechanism and Machine Theory*, 1975, 10(6): 467–478.
[17] F L Litvin, A Fuentes, G Ignacio, et al. *Noncircular gears: design and generation*. New York: Cambridge University Press, 2009.
[18] D W Liu, Y B Ba, T Z Ren, et al. Flow fluctuation abatement of high-order elliptical gear pump by external noncircular gear drive. *Mechanism and Machine Theory*, 2019, 134: 338–348.
[19] X Zhang, S W Fan. Study on the pitch curve with minimal rotary inertia for the noncircular gears. *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, 2018, 232(15): 2666–2673.
[20] M Li, T Shi, J Yang, et al. Realizing nonlinear springs through noncircular planetary gears. *Mechanism and Machine Theory*, 2021, 156: 104151.
[21] J Y Tang, Z W Wang, D C Lei, et al. Study on the relationship between load and gear mesh stiffness, contact ratio. *Journal of Mechanical Transmission*, 2014, 38(6): 1–4. (in Chinese)
[22] F Y Zheng, L Hua, X H Han, et al. Synthesis of indexing mechanisms with non-circular gears. *Mechanism and Machine Theory*, 2016, 105: 108–128.
[23] F Y Zheng, X H Han, Lin Hua, et al. Design and manufacture of new type of non-circular cylindrical gear generated by face-milling method. *Mechanism and Machine Theory*, 2018, 122: 326–346.
[24] P Alexandru, D MacAveiu, Cătălin Alexandru, et al. A gear with translational wheel for a variable transmission ratio and applications to steering box. *Mechanism and Machine Theory*, 2012, 52: 267–276.
[25] Z C Huang, Z H Lan. Design of transmission ratio function of noncircular gear of closed pitch curve. *Journal of Mechanical Transmission*, 2011, 11: 34–37. (in Chinese)