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Over-Constraints and a Unified Mobility Method for General Spatial Mechanisms Part 2: Application of the Principle

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Abstract: The pre-research on mobility analysis presented a unified-mobility formula and a methodology based on reciprocal screw theory by HUANG, which focused on classical and modern parallel mechanisms. However its range of application needs to further extend to general multi-loop spatial mechanism. This kind of mechanism is not only more complex in structure but also with strong motion coupling among loops, making the mobility analysis even more complicated, and the relevant research has long been ignored. It is focused on how to apply the new principle for general spatial mechanism to those various multi-loop spatial mechanisms, and some new meaningful knowledge is further found. Several typical examples of the general multi-loop spatial mechanisms with motion couple even strong motion couple are considered. These spatial mechanisms include different closing way: over-constraint appearing in rigid closure, in movable closure, and in dynamic closure as well; these examples also include two different new methods to solve this kind of issue: the way to recognize over-constraints by analyzing relative movement between two connected links and by constructing a virtual loop to recognize over-constraints. In addition, over-constraint determination tabulation is brought to analyze the motion couple. The researches above are all based upon the screw theory. All these multi-loop spatial mechanisms with different kinds of structures can completely be solved by following the directions and examples, and the new mobility theory based on the screw theory is also proved to be valid. This study not only enriches and develops the theory and makes the theory more universal, but also has a special meaning for innovation in mechanical engineering.

Keywords: mobility analysis, over-constraint, multi-loop spatial mechanism, motion coupling, screw theory

1 Introduction

It is well known that the spatial mechanism, which can offer advantages in terms of flexibleness and diverseness, compact-sized, and dynamic performances etc., has been applied in many fields[1]. For a new mechanism, mobility calculation is the first consideration in the kinematic and the dynamic modelling of mechanisms. And it has been an important topic in more than 150 years. In history, sustained efforts are made and various approaches are derived[2–6].

In Refs. [7–8], a unified-mobility formula based on reciprocal screw theory has been presented and developed. Based on the methodology, it is pointed that the reason of mobility difficulty is the over-constraint in mechanism, and those documents focused only on classical and modern parallel mechanisms. The purpose of this paper is to further extend the theory to more general multi-loop spatial mechanisms. The parallel mechanism is multi-loop, compared with which, the structure of the general multi-loop spatial mechanism is more complex and the mobility analysis is more complicated and troublesome[9–11]. It is not only because their loops are not parallel but also their independent loops in topology are not independent in kinematics, which is due to their irregular loop-closures as well as the appearance of the motion coupling even strong motion coupling between their “independent loops” sometimes[12–14]. However, these issues have not even been studied systematically before.

In order to solve the issue, some basic theoretical principles are given in Ref. [15], where the reason of over-constraint appearing is pointed as the closure of kinematic chains. And the closure can be classified as three cases: rigid closure, movable closure and dynamic closure. And the motion couple is also analyzed, and two ways are proposed to solve the couple issue including “recognizing over-constraints by analyzing relative movement” and “recognizing over-constraints by virtual loop”. All these form a systematic mobility principle for the general multi-loop spatial mechanism.

On the one hand, it needs to check the above-mentioned mobility principle whether to be correct or not, and needs to show how to apply the systematic new principle to analyze the different situations including the different

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closure ways and the different motion-couple grade; on the other hand, there are plentiful and various kinds of types for the general multi-loop spatial mechanisms, and their solution needs to be carefully researched further. In addition, analyzing the over-constraint with an "over-constraint determination table" is introduced here and it also is a feasible and simple way. It can make the concept of over-constraint clear and easy to understand. By application of this method as well as the screw theory and through the research process, not only some new regular patterns are found, but also it could conclude and demonstrate the complex spatial mobility issue can be transformed to be very simple and clear.

2 Single-loop Spatial Mechanism

Similar to mobility analysis of other mechanisms, for the general spatial ones the over-constraint, μ, needs to be determined firstly based on the theorem mentioned in Ref. [15], which indicates that for single-loop mechanisms the number of over-constraints is just that of the independent reciprocal screws. After recognizing over-constraints the mobility of the mechanism can be calculated by the unified-mobility formula[7–8] expressed as follows:

$$M = 6\left(n-g-1\right)+\sum f_i+\mu\,, \tag{1}$$

where M denotes the mobility of the mechanism; n the number of links including the frame; g the number of the kinematic pairs; f_i the freedom of the ith kinematic pair; and μ the total number of over-constraints of the mechanism. It is named "unified-mobility formula".

Fig. 1 shows a spatial four-bar linkage mechanism $ABCDA$ with two spherical pairs[16]. For its mobility analysis, it has no doubt that it is necessary to estimate whether and how many over-constraints exist in the mechanism firstly.

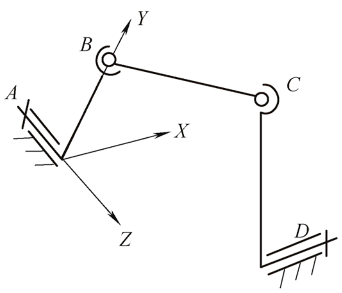

Fig. 1. Spatial four-bar linkage mechanism with double spherical pairs

To express its screw system a coordinate system A-XYZ is established in Fig. 1, where Z-axis is along the direction of the revolute pair A, Y-axis along the link AB, and X-axis accordance with the right-hand rule. Considering a spherical pair has three twist screws, for the whole mechanism the twist system including eight screws can be written as follows:

$$\begin{aligned}
\$_A &= \left(0 \quad 0 \quad 1;\quad 0 \quad 0 \quad 0\right),\\
\$_{B1} &= \left(1 \quad 0 \quad 0;\quad 0 \quad 0 \quad f_2\right),\\
\$_{B2} &= \left(0 \quad 1 \quad 0;\quad 0 \quad 0 \quad 0\right),\\
\$_{B3} &= \left(0 \quad 0 \quad 1;\quad d_4 \quad 0 \quad 0\right),\\
\$_{C1} &= \left(1 \quad 0 \quad 0;\quad 0 \quad e_5 \quad f_5\right),\\
\$_{C2} &= \left(0 \quad 1 \quad 0;\quad d_6 \quad 0 \quad f_6\right),\\
\$_{C3} &= \left(0 \quad 0 \quad 1;\quad d_7 \quad e_7 \quad 0\right),\\
\$_D &= \left(a_8 \quad b_8 \quad c_8;\quad d_8 \quad e_8 \quad f_8\right),
\end{aligned} \tag{2}$$

where the subscript in the lower right corner of each screw denotes the corresponding kinematic pair. Obviously, the eight screws are linearly dependent and six of them are independent, which indicates the rank of the screw system is six. So there is no constraint screw and no over-constraint, i.e. $\mu = 0$. Based on Eq. (1), there is

$$M = 6\left(n-g-1\right)+\sum f_i+\mu = 6(4-4-1)+8+0 = 2. \tag{3}$$

The mobility of the mechanism is two, including one local freedom of link BC, which can rotate freely around its own axis. That occurrence is because of a linearly dependence of the screw system in Eq. (2).

3 Spatial Double-loop Mechanism

3.1 General spatial double-loop mechanism

This mechanism as Fig. 2 shows contains thirteen links and fourteen kinematic pairs, which forms two loops. The first loop, $ABC\cdots GA$, has seven revolute pairs, where four of them are parallel to each other, and the other three are also parallel to each other. But the axes of the two group screws are perpendicular to each other. The second loop, $CDEH\cdots NC$, has ten kinematic pairs, but only seven of them belong to the second loop independently, and the axes H, I, J are parallel to that of L, M, N.

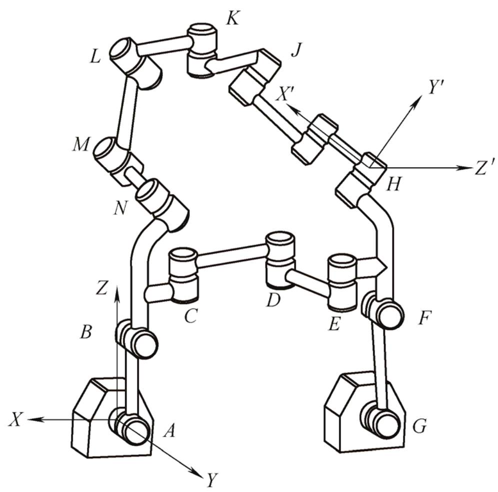

Fig. 2. Spatial thirteen-links mechanism with double loops

(1) For the first loop, under the coordinate system *A-XYZ*, four axes of kinematic pairs *A, B, F, G* are all along *Y*-axis and the axes of other three pairs are along *Z*-axis. The corresponding screws are as follows:

$$\begin{aligned}
\$_A &= (0 \quad 1 \quad 0;\quad 0 \quad 0 \quad 0),\\
\$_B &= (0 \quad 1 \quad 0;\quad d_2 \quad 0 \quad f_2),\\
\$_F &= (0 \quad 1 \quad 0;\quad d_3 \quad 0 \quad f_3),\\
\$_G &= (0 \quad 1 \quad 0;\quad 0 \quad 0 \quad f_4),\\
\$_C &= (0 \quad 0 \quad 1;\quad d_5 \quad e_5 \quad 0),\\
\$_D &= (0 \quad 0 \quad 1;\quad d_6 \quad e_6 \quad 0),\\
\$_E &= (0 \quad 0 \quad 1;\quad d_7 \quad e_7 \quad 0).
\end{aligned} \tag{4}$$

The rank of the screw system is only five and there is one reciprocal screw:

$$\$_1^r = (0 \quad 0 \quad 0;\quad 1 \quad 0 \quad 0). \tag{5}$$

Therefore there is an over-constraint, $\mu_1 = 1$. The mobility of the loop can be calculated as follows:

$$M_1 = 6(n-g-1) + \sum f_i + \mu = 6(7-7-1) + 7 + 1 = 2. \tag{6}$$

It indicates the mobility of the first loop is two.

(2) For the second loop, $CDEH\cdots NC$, the number of the independent kinematic pairs, (new pairs with respect to that in the old loop) in the loop is seven, and the axis of *K* is not parallel to any of the other axes of the new pairs. For the new coordinate system *H-X'Y'Z'*, it can be achieved as follows:

$$\begin{aligned}
\$_H &= (0 \quad 1 \quad 0;\quad 0 \quad 0 \quad 0),\\
\$_I &= (0 \quad 1 \quad 0;\quad 0 \quad 0 \quad f_9),\\
\$_J &= (0 \quad 1 \quad 0;\quad d_{10} \quad 0 \quad f_{10}),\\
\$_K &= (a_{11} \quad b_{11} \quad c_{11};\quad d_{11} \quad e_{11} \quad f_{11}),\\
\$_L &= (a_{12} \quad b_{12} \quad c_{12};\quad d_{12} \quad e_{12} \quad f_{12}),\\
\$_M &= (a_{12} \quad b_{12} \quad c_{12};\quad d_{13} \quad e_{13} \quad f_{13}),\\
\$_N &= (a_{12} \quad b_{12} \quad c_{12};\quad d_{14} \quad e_{14} \quad f_{14}).
\end{aligned} \tag{7}$$

It is very clear that the seven screws are linearly dependent, and their rank is six. Since its rank is six and when adding the serial chain with these seven screws to the first loop, it is clearly impossible to bring into any over-constraint to the linkage, that means, $\mu_2 = 0$

The whole mechanism has thirteen links and fourteen pairs, and the total over-constraints are

$$\mu = \mu_1 + \mu_2 = 1. \tag{8}$$

By mobility calculating, it could be concluded that

$$M = 6(n-g-1) + \sum f_i + \mu = 6(13-14-1) + 14 + 1 = 3. \tag{9}$$

The mobility of this mechanism is three.

From this double-loop example it is found that there is neither any kinematic couple between the two loops nor any new over-constraint in the new loop. That is because only the rank of new screw system for these "new pairs" of the new loop is up to six.

3.2 Recognizing over-constraints by analyzing relative movement

Fig. 3(a) shows a mechanical coupler with double sliding rods which can transmit a motion between two shafts with a 90° orthogonal angle[17]. Axis of disk *A* is input and that of *B* is output, both disks are connected with the frame by two *R* pairs. There are five links, two revolute pairs and four cylindrical pairs. The two cylindrical links connect with the driving and passive dicks in the mechanism.

In the problem, there is a sub-mechanism, *CDEF*, and it brings real troublesome to find out its over-constraint because of the complicated structure. Firstly the sub-mechanism *CDEF* is taken into consideration.

(1) Estimate the over-constraint of sub-loop *CDEFC*

As shown in Fig. 3(b), a coordinate system *A-XYZ* is set, in which *X*-axis and *Y*-axis are parallel to the two sides of the right-angle link, respectively, and the origin point *A* locates at the center of circle *CF*. The screw system can be obtained:

$$\begin{aligned}
\$_{C1} &= (1 \quad 0 \quad 0;\quad 0 \quad e_1 \quad f_1),\\
\$_{C2} &= (0 \quad 0 \quad 0;\quad 1 \quad 0 \quad 0),\\
\$_{D1} &= (0 \quad 1 \quad 0;\quad d_3 \quad 0 \quad f_3),\\
\$_{D2} &= (0 \quad 0 \quad 0;\quad 0 \quad 1 \quad 0),\\
\$_{E1} &= (0 \quad 1 \quad 0;\quad d_5 \quad 0 \quad f_5),\\
\$_{E2} &= (0 \quad 0 \quad 0;\quad 0 \quad 1 \quad 0),\\
\$_{F1} &= (1 \quad 0 \quad 0;\quad 0 \quad e_7 \quad f_7),\\
\$_{F2} &= (0 \quad 0 \quad 0;\quad 1 \quad 0 \quad 0),
\end{aligned} \tag{10}$$

which can be simplified as

$$\begin{aligned}
\$_{C1} &= (1 \quad 0 \quad 0;\quad 0 \quad e_1 \quad f_1),\\
\$_{C2} &= (0 \quad 0 \quad 0;\quad 1 \quad 0 \quad 0),\\
\$_{D1} &= (0 \quad 1 \quad 0;\quad d_3 \quad 0 \quad f_3),\\
\$_{D2} &= (0 \quad 0 \quad 0;\quad 0 \quad 1 \quad 0),\\
\$_{F1} &= (1 \quad 0 \quad 0;\quad 0 \quad e_7 \quad f_7).
\end{aligned} \tag{11}$$

The rank of the screw system is five and it has a reciprocal screw expressed as

$$\$^{\mathrm{r}} = (0 \quad 0 \quad 0;\quad 0 \quad 0 \quad 1). \tag{12}$$

It is a constraint couple limiting a rotation about Z-axis. There is one over-constraint in the sub-mechanism, $\mu_1 = 1$. Consequently, we have n=4, g=4 and $\mu_1 = 1$ in this sub-mechanism, and its mobility is

$$M_1 = 6(n-g-1)+\sum f_i + \mu_1 = 6(4-4-1)+8+1=3. \tag{13}$$

From this analysis, taking *CF* as the frame, the output link *ED* has three degree of freedoms(DOFs), including two translations along *X*- and *Y*-axes, and a rotation around *X*-axis.

(2) Estimate the over-constraint of the second closed-loop

The mechanical coupler, as shown in Fig. 3(a), will be obtained by adding two revolute pairs *A* and *B* in front and at back of the sub-mechanism, respectively. Meanwhile the second closed loop, Fig. 3(c), will be formed, and the new over-constraints will appear when reclosing the revolute pair *B* with the frame.

Here it needs to check the relative motions between the two elements in closure to determine the over-constraints by using “method to recognize over-constraints by analyzing relative movement”.

For the spatial mechanism the over-constraint must be analyzed from six aspects including three translations along and three rotations around *X*-, *Y*- and *Z*-axes, respectively. Each of which should be checked to determine which constraint is real and which one is virtual.

That is to say, to check the over-constraints in closing, the movements of both the end links of two chains should be analyzed and compared, and the constraints produced in closure may be obtained. That is to say, from which the constraint to be real or virtual could also be determined in this way. It is just the basic idea of the “method of analyzing the relative movements between closed elements” for determination of over-constraints.

Connecting pair *A* of the sub-chain to the base, and link *ED* becomes the end-link of the open chain. Before further closing the revolute pair *B*, the end link *ED* of the open chain *ACFED* has four DOFs including two translations along *X*- and *Y*-axes, two rotations around *X*- and *Y*-axes, that is because pair *A* has been added and connected to the base and one more DOF is added to the sub-loop. The four screws for the four mobilities form a virtual-open chain waiting for reclosure, Fig. 3(c) Correspondingly, after closing the virtual chain to the base by using the revolute pair *B*, the virtual loop *AMPQB* is formed, as shown in Fig. 3(d), where three virtual pairs *M*, *P, Q*, denote the three screws for the three DOFs in Eq. (11). The reclosure pair *B* will bring into five constraints to the end link *ED*. To analyze how many and what kinds of constraints are there in closure, here a method named “over-constraint determination tabulation method”, Table 1, will be applied.

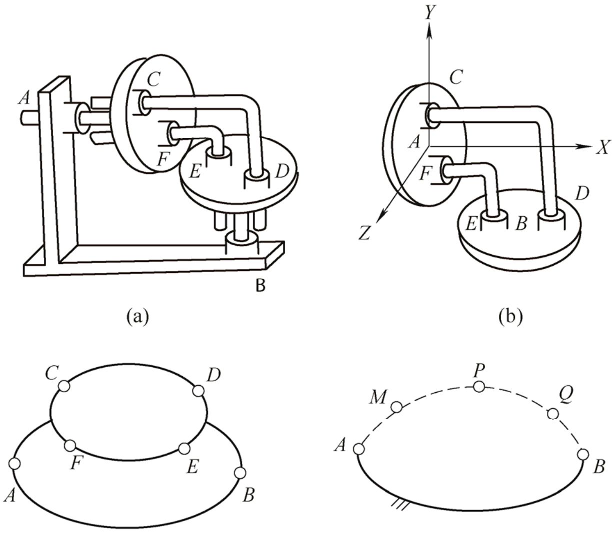

Fig. 3. Couplings with double right angle slider

Table 1 makes some comparisons on the relative movements and constraints before and after the closing, and determines the over-constraints.

Table 1. Over-constraint determination table

Item	Translational mobility			Revolute mobility		
	X	*Y*	*Z*	*X*	*Y*	*Z*
Mobility of *DE*	Enable	Enable	Restricted	Enable	Restricted	Restricted
Revolute pair *B*	Restricted	Restricted	Restricted	Restricted	Permit	Restricted
Comparing	Real constraint	Real constraint	Virtual constraint	Real constraint	No constraint	Virtual constraint

In analysis, if one certain DOF of the end-link of the open chain exists before closing, while it is restricted by the closure pair in closing, it means the closure pair brings a real constraint to the end link of the open chain. If one certain DOF is allowed before closing and not restricted by the closure pair in closing, or one DOF is restricted before closing but allowed by the closure pair, they all mean that the closure pair brings no constraint to the end link. Only when a constraint is brought to the open chain before closing and then also produced by the closure pair once again, it will form a repeated constraint, which is just the “virtual constraint”.

Therefore, it can be seen according to the comparison on the relative movements and constraints before and after closing the joint *B*, the translation DOFs along *X*- and *Y*-axes, the rotation around *X*-axis are all constrained and they are real constraints; and the rotation around Y-axis is not constrained; while the translation along Z-axis and the rotation around Z-axis just belong to the virtual constraints which is being searched. So that when the pair B is closed there are two over-constraints, i.e., $\mu_2 = 2$.

For the whole mechanism, the total number of over constraints is $\mu = \mu_1 + \mu_2 = 3$. In addition, the mobility can be calculated by

$$M = 6(n-g-1) + \sum f_i + \mu = 6(5-6-1) + 10 + 3 = 1. \tag{14}$$

For general multi-loop spatial mechanisms, to obtain the over-constraint of a close-loop it must adopt to open it and form a open chain by cutting a link or a joint, and then check the mobility number and property of the two elements before reclosure based on the screw theory. Note that if one of the two closure elements is a frame, it is in rigid closure. In this case, one of the two elements is static and it has no DOF, all the constraints applied to the end-link of the open chain are just the over-constraints in closure. For dynamical closure, both of the two closure elements are active, the over-constraints should be determined by checking the relative movements and analyzing the constraints between the two elements in closure. As mentioned before, there are three forms for closure: rigid closure, active closure, and dynamic closure, in which the over-constraints should be identified according to different circumstances.

For the second or other latter closed loops of the mechanism, the over-constraints should be also dealt with based on “method of analyzing the relative movements between closed elements” in the same way, that is because the motion couple may appear and educe over-constraint. From this example complex problems are exposed, and the over-constraints can be solved by using the aid of “over-constraint determination tabulation” and it makes the concept very clear.

3.3 Spatial mechanism with dynamic closure

Fig. 4 shows a mechanism which can transform revolution to translation[18]. It is also a spatial double-loop mechanism, and its link CF with square section locates into a groove of link ED. The mechanism has five links and six pairs(here the axes of pairs A and D are collinear). O and F are revolute pairs, G cylindrical, and C spherical. Link CF has three DOFs and pair E is a planar pair also with three DOFs, Fig. 4(b). The mechanism has two loops, as shown in Fig. 4(c). The first loop consists of pairs O, C, E and D, and the second one contains four pairs, D, E, F, and G.

It also needs to check the over-constraint firstly before mobility calculation. A coordinate system C-XYZ, where the origin point locates at point C is built in Fig. 4(a). Consider the independent pairs in the second loop is less than six, and it does not need to recognize the over-constraint, i.e. $\mu_1 = 0$.

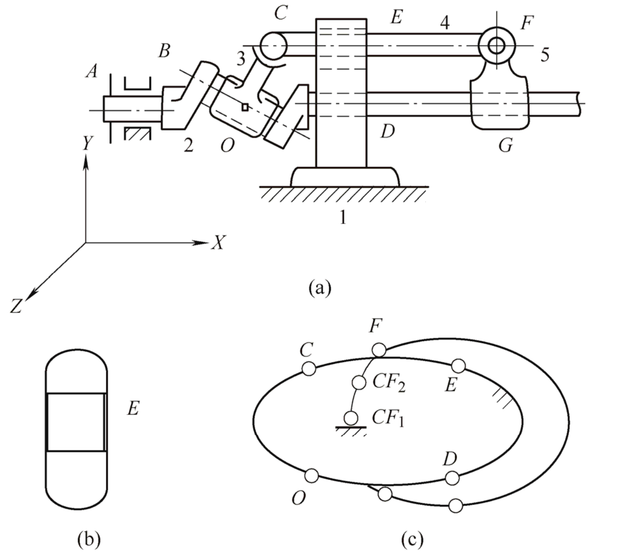

Fig. 4. Revolute-translation linkage

3.3.1 Basic loop OCEDO

There are eight screws as follows:

$$\begin{aligned}
\$_D &= (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad f_1),\\
\$_O &= (a_2 \quad b_2 \quad 0; \quad 0 \quad 0 \quad f_2),\\
\$_{C1} &= (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0),\\
\$_{C2} &= (0 \quad 1 \quad 0; \quad 0 \quad 0 \quad 0),\\
\$_{C3} &= (0 \quad 0 \quad 1; \quad 0 \quad 0 \quad 0),\\
\$_{E1} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0),\\
\$_{E2} &= (0 \quad 0 \quad 0; \quad 0 \quad 1 \quad 0),\\
\$_{E3} &= (0 \quad 0 \quad 1; \quad 0 \quad e_8 \quad 0).
\end{aligned} \tag{15}$$

As what is known, $\mu_1 = 0$, and the mobility of the first loop is

$$M_1 = 6(n-g-1) + \sum f_i + \mu_1 = 6(4-4-1) + 8 + 0 = 2. \tag{16}$$

The first loop has two DOFs and the corresponding twist screws of link CF are

$$\begin{aligned}
\$_{CF1} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0),\\
\$_{CF2} &= (0 \quad 0 \quad 1; \quad 0 \quad e_2 \quad 0).
\end{aligned} \tag{17}$$

3.3.2 Second loop

Since the new loop (the 2nd loop) has more than one link, FG, and two pairs, F and G to the first loop, the number of new twist screws of the new pairs is less than six, and their rank is also less than six. In this case it should check whether or not to bring any over-constraint to the mechanism.

Due to the fact that points F and G both are in moving, the closure here is in dynamic closure, which is more complex.

Here link FG has three DOFs, including translations along X- and Y-axes, as well as rotation about Z-axis around point F before closure. Since the rank of new screw system for these "new pairs" of the new loop is less than six and the motion couple may happen, the over-constraint should be checked. For simple, the virtual loop way may be applied.

3.3.3 Virtual loop way

The virtual loop way is taken to analyze the second loop. How many and which kinematic pairs belong to the virtual loop? They include the twist screw for pair F, the two screws for two mobility of link CF, Eq. (17), the two screws for pair G, as well as the screw for link DG, and form the virtual loop, Fig. 4(c). The total number is six. The corresponding screw system is as follows:

$$\begin{aligned} \$_{CF1} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0), \\ \$_{CF2} &= (0 \quad 0 \quad 1; \quad 0 \quad e_2 \quad 0), \\ \$_{F} &= (0 \quad 0 \quad 1; \quad 0 \quad e_3 \quad 0), \\ \$_{G1} &= (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad f_4), \\ \$_{G2} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0), \\ \$_{D} &= (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad f_4), \end{aligned} \tag{18}$$

where screws CF_1, CF_2 and G_1 are virtual pairs. As the virtual loop has six screws and their rank is only four, which indicates there are two over-constraints, $\mu_2 = 2$.

The whole mechanism has five links, and six pairs with eleven motilities, and the total over-constraints are $\mu = \mu_1 + \mu_2 = 2$, then we have

$$M = 6(n - g - 1) + \sum f_i + \mu = 6(5 - 6 - 1) + 11 + 2 = 1. \tag{19}$$

The virtual loop way is simpler than the former.

3.4 A complex spatial mechanism

Fig. 5(a) shows a hook-thread mechanism for a sewing machine. The crank AB drives the rocker CDH via a link rod BC involving two spherical pairs. When connecting points G and H are also two spherical pairs, the output point I in link FG will be driven to achieve a desired trajectory.

Firstly, the analysis of the over-constraint is needed before calculating the mobility of this mechanism. This mechanism is a complex multi-loop spatial one. To obtain the over-constraint, such complex problem should be simplified and separated into some sub-parts as far as possible based on the existent knowledge.

For this mechanism, it can be considered to have three sub-mechanisms: A four-bar linkage $ABCD$ with double spherical pairs, a spatial guide-bar mechanism $AEFI$, and also another four-bar linkage $FGHD$ composed with two links DH and FG which are also connected by two spherical pairs G and H. Therefore, the complex mechanism could be decomposed into three separate sub-mechanisms, as shown in Fig. 5(b). The process will be broke down into four steps.

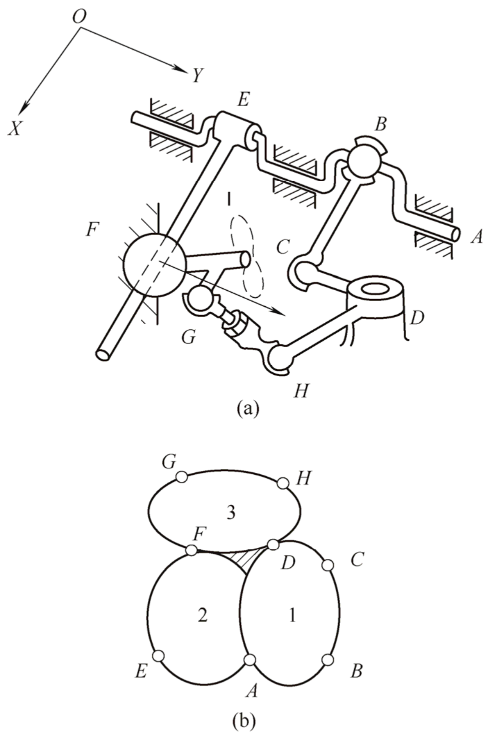

Fig. 5. Hook-thread mechanism of a sewing machine

(1) Estimate the over-constraint of $ABCDA$. According to the analysis above, the four-bar linkage mechanism $ABCDA$ with double spherical pairs has no over- constraint, $\mu_1 = 0$, and has one DOF besides a local DOF.

(2) Estimate the over-constraint of $AEFIA$. The spatial guide-bar mechanism $AEFIA$ consists of crank AE, a guide link EF and a spherical link FG. It is a single-loop mechanism with four links and four kinematic pairs. For the second loop since the crank passes through the axis A and identical with that of the first loop, there is no any kinematic couple between the two loops and the mobility formula can be applied directly. In the coordinate system O-XYZ, the center of A-axis is chosen as the origin point, and X-axis is along EF and Y-axis along A-axis. Based on the screw theory, the screw system of the sub-loop is as follows:

$$\begin{aligned} \$_{A} &= (0 \quad 1 \quad 0; \quad 0 \quad 0 \quad 0), \\ \$_{E} &= (0 \quad 1 \quad 0; \quad d_2 \quad 0 \quad f_2), \\ \$_{F1} &= (1 \quad 0 \quad 0; \quad 0 \quad e_3 \quad f_3), \\ \$_{F2} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0), \\ \$_{F3} &= (1 \quad 0 \quad 0; \quad 0 \quad e_3 \quad f_3), \\ \$_{F4} &= (0 \quad 1 \quad 0; \quad d_6 \quad 0 \quad f_6), \\ \$_{F5} &= (0 \quad 0 \quad 1; \quad d_7 \quad e_7 \quad 0). \end{aligned} \tag{20}$$

The seven screws are linearly dependent and only five of them are independent. They have one reciprocal screw, and the reciprocal screw is

$$\boldsymbol{\$}^r = \begin{pmatrix} 0 & 1 & 0; & -e_3 & 0 & -e_7 \end{pmatrix}. \tag{21}$$

It is a constraint force parallel to Y-axis and passing through point F. It is denoted by a vector arrow in Fig. 5.

Since the sub-mechanism AEF is a single loop one, the number of reciprocal screw is just its over-constraint, that is $\mu_2 = 1$.

(3) Sub-mechanism $FGHDF$. The $FGHDF$ has four links. There are three spherical pairs(F, G and H) and one revolute pair D in the sub-mechanism. However, in fact, as link EF passes through the sphere center F and limits its two rotational freedoms, only one rotation freedom is left. That is to say, the four-bar linkage $FGHDF$ can also be considered as a mechanism with only two spherical pairs like the mechanism $ABCDA$. Since $FGHDF$ has the same mobility with $ABCDA$, there is no any over-constraint, $\mu_3 = 0$. And it has one DOF besides a local one. Considering the input link of the sub-loop is connected to the base directly, the motion couple also does not happen.

(4) For the whole mechanism. There are seven links and nine joints including five spherical pairs, one cylindrical pair and three revolute ones, and the number of over-constraints is $\mu = \mu_1 + \mu_2 + \mu_3 = 1$ in total. Then under the unified-mobility criterion, it is summed as

$$M = 6(n - g - 1) + \sum f_i + \mu = 6(7 - 9 - 1) + 20 + 1 = 3. \tag{22}$$

There are three DOFs in the mechanism including two local DOFs, where links BC and GH can rotate freely.

Note that for a complicated mechanism, it should be simplified firstly to break down the difficult problems into some easier ones, which is a common and nice choice.

From this example it is found that although the linkage has three loops, there is no any motion couple among them and this property has been expressed in three-loop graph, as shown in Fig 5(b). Therefore, for kinematic analysis the loop graph should be considered and analyzed.

4 Strong Coupling Multi-Loop Spatial Mechanisms

For the example of Altmann No. 35[18], how to analyze the mobility of the strong coupling multi-loop spatial mechanism is shown hereby. The structure diagram and several parts of this mechanism are shown in Fig. 6(a). Fig. 6(b) is its kinematic diagram.

Two cubic blocks c_1 and c_2 are installed on the input and output shafts b_1 and b_2 by two revolute pairs, respectively, Fig. 6(a) and Fig. 6(b), and meanwhile c_1 connects with c_2 by a planar kinematic pair. On the top and bottom of c_1 and c_2, another thin block e with cylindrical concave surface is installed, respectively. Thus there are total four thin blocks which connect with two tablets CE and JH, and on each tablet there are two half-cylindrical convex. Each tablet joins two thin blocks e by two cylindrical pairs, C, E or J, H, and it also joins with frame by top or lower two revolute pairs D or I.

Here concave e_i and block c_i could be regarded as only one link since between them it is impossible to give rise to any relative motion. In addition, it can be seen that this transmission system keeps strictly symmetrical in geometry.

According to the schematic diagram of the mechanism in Fig. 6(b), there are seven links and eleven kinematic pairs including four revolute pairs A, D, G and I, six cylindrical pairs B, C, E, F, H, and J, and a planar pair K which has three DOFs. Obviously, the result is wrong if the over-constraints are ignored:

$$M = 6(n - g - 1) + \sum f_i + \mu = 6(7 - 11 - 1) + 19 + 0 = -11. \tag{23}$$

Fig. 6(c) shows a schematic diagram with only revolute pairs, in which the four dotted lines AD, AI, GD and GI also denote the same frame. For this complex multi-loop mechanism, Euler's Theorem can be adopted to obtain the number of the independent loops of the mechanism as

$$l = g - n + 1 = 11 - 7 + 1 = 5. \tag{24}$$

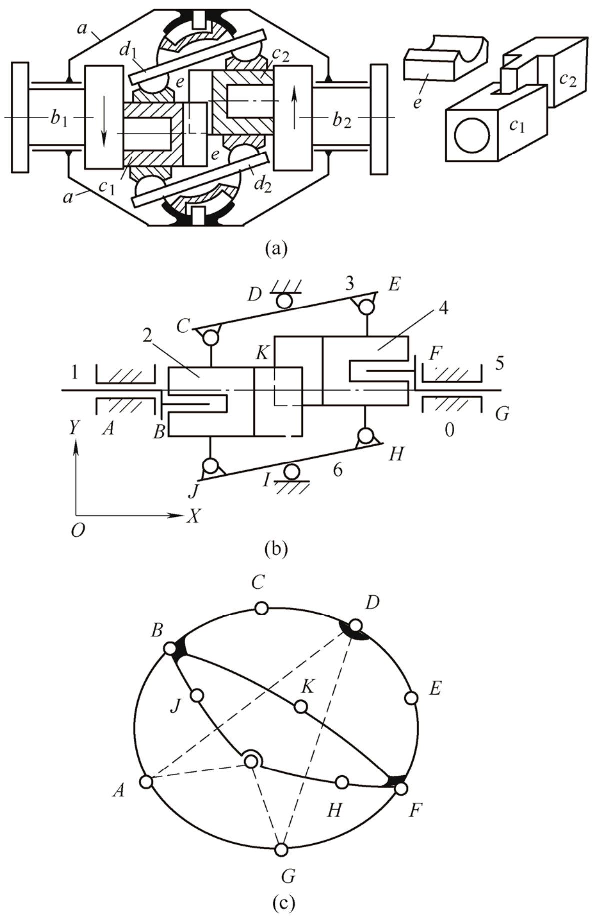

Fig. 6. Altmann No. 35 transmission gear

That indicates there are five independent loops in this mechanism including the first loop *ABKFGA*, the second one *ABCDA*, the third *DEFGD*, the fourth *GFHIG* and the fifth *ABJIA*. The over-constraints of this mechanism may exist in each closed loop. Therefore, the constraints in each loop should be analyzed and recognized respectively and then the over-constraints of this mechanism can be obtained. So the following steps are taken to solve this difficult problem.

4.1 Sub-mechanism *ABKFGA*

The coordinate system is shown in Fig. 6(b), in which X-axis is along the center axis, and Z-axis is perpendicular to the paper plane. Then, the screw system of the five kinematic pairs in loop *ABKFGA* can be expressed by the following nine screws:

$$\begin{aligned}
\$_A &= (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0), \\
\$_{B1} &= (1 \quad 0 \quad 0; \quad 0 \quad e_2 \quad f_2), \\
\$_{B2} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0), \\
\$_{K1} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0), \\
\$_{K2} &= (0 \quad 0 \quad 0; \quad 0 \quad 1 \quad 0), \\
\$_{K3} &= (0 \quad 0 \quad 1; \quad 0 \quad 0 \quad 0), \\
\$_{F1} &= (1 \quad 0 \quad 0; \quad 0 \quad e_7 \quad f_7), \\
\$_{F2} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0), \\
\$_G &= (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0).
\end{aligned} \tag{25}$$

The nine screws are linearly dependent and only five screws are independent. Thus, one over-constraint can be obtained, $\mu_1 = 1$. In addition, the degree of freedom can be calculated, and is obtained as

$$M_1 = 6(n-g-1) + \sum f_i + \mu_1 = 6(5-5-1) + 9 + 1 = 4. \tag{26}$$

This closed loop has four DOFs, three of them are local DOFs, including two translations along X-axis of the two rotator blocks, independently, and one translating along Z-axis simultaneously and instantaneously of the two rotator blocks. It indicates the output mobility of this loop is only one.

4.2 Sub-mechanism *ABCDA*

To calculate the over-constraint of the second loop *ABCDA*, the motion coupling between the first two loops must be taken into consideration. Here, the method of virtual loop is applied. Firstly, the mobility number and corresponding property of link 2(*BK*) should be firstly solved. Link 2 is just the link included in the first two loops simultaneously and therefore the kinematic coupling may happen. When obtaining the twist screws of link *BK*, and adding the corresponding screws of kinematic pairs *C* and *D*, a new screw system is formed. In the kinematic chain including the new screw system, not all kinematic pairs are actual ones. That is why it is named “virtual closed-loop kinematic chain”, and the method of virtual loop is applied to deal with it.

(1) Determine the screw of link *BK*. Above all, the degree of freedom number and property of link 2(*BK*) in the first closed-loop should be worked out, and all the pairs must be expressed as twist screws as well. To do this, the *ABKFGA* should be regarded as a parallel mechanism with two limbs *AB* and *KFG*. Link 2 is assumed as its output link. The two limbs have three and six screws, which are just the first three screws and the latter six ones in Eq. (25) and their corresponding reciprocal screws are shown respectively as follows:

$$\begin{aligned}
\$_A &= (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0), \\
\$_{B1} &= (1 \quad 0 \quad 0; \quad 0 \quad e_2 \quad f_2), \\
\$_{B2} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0);
\end{aligned} \tag{27}$$

$$\begin{aligned}
\$_{11}^r &= (0 \quad 0 \quad 0; \quad 0 \quad 1 \quad 0), \\
\$_{12}^r &= (0 \quad 0 \quad 0; \quad 0 \quad 0 \quad 1), \\
\$_{13}^r &= (0 \quad -f_2 \quad e_2; \quad 0 \quad 0 \quad 0);
\end{aligned} \tag{28}$$

$$\begin{aligned}
\$_{K1} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0), \\
\$_{K2} &= (0 \quad 0 \quad 0; \quad 0 \quad 1 \quad 0), \\
\$_{K3} &= (0 \quad 0 \quad 1; \quad 0 \quad 0 \quad 0), \\
\$_{F1} &= (1 \quad 0 \quad 0; \quad 0 \quad e_7 \quad f_7), \\
\$_{F2} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0), \\
\$_G &= (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0);
\end{aligned} \tag{29}$$

$$\$_{21}^r = (0 \quad 0 \quad 0; \quad 0 \quad 1 \quad 0). \tag{30}$$

These two limbs have four reciprocal screws in total which all act on its moving platform *BK*. And their secondary reciprocal screws will express the motion of link *BK*. Since the four screws are linearly dependent, there are only three independent reciprocal screws, which represent three twist screws of *BK* in the first loop:

$$\begin{aligned}
\$_{BK1}^{rr} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0), \\
\$_{BK2}^{rr} &= (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0), \\
\$_{BK3}^{rr} &= (0 \quad 0 \quad 0; \quad 0 \quad e_2 \quad f_2).
\end{aligned} \tag{31}$$

(2) Establish the screw system of virtual-loop chain, *BCD*. The screws in this system include the screws of *BK* shown in Eq. (31), along with another three screws of kinematic pairs *C* and *D*, therefore

$$\begin{aligned}
\$_{BK1} &= (1 \quad 0 \quad 0; \quad 0 \quad 0 \quad 0), \\
\$_{BK2} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0), \\
\$_{BK3} &= (0 \quad 0 \quad 0; \quad 0 \quad e_2 \quad f_2), \\
\$_{C1} &= (0 \quad 0 \quad 1; \quad d_4 \quad e_4 \quad 0), \\
\$_{C2} &= (0 \quad 0 \quad 0; \quad 0 \quad 0 \quad 1), \\
\$_{D} &= (0 \quad 0 \quad 1; \quad d_6 \quad e_6 \quad 0).
\end{aligned} \tag{32}$$

The six screws are linearly dependent and its rank is five. It has one reciprocal screw:

$$\$^r_{21} = (0 \quad 0 \quad 0; \quad 0 \quad 1 \quad 0). \tag{33}$$

Thus, the virtual loop *BCD* produces one over-constraint to the mechanism, $\mu_2 = 1$.

4.3 Sub-mechanism *DEFGD*

The third reclosure happens at point *E* of the third loop, which is the common joint of the two mobile links, link *DE* and link 4(*FK*). The motions of these two links should be determined firstly, and then, virtual closed-loop may be built as well as the over-constraint in the closed loop may be checked.

For kinematic chain *DE*, it has two kinematic pairs containing three screws. However, the factor which should be paid special attention to is the strict geometry symmetry of the mechanism both in the structure and size, and because of this, point *E* in link *CDE* should be definitely located in a place where it can exactly contact with point *E* of link *EF*, then from this point of view, point *E* can be treated as a fixed point.

Next step is to analyze the link 4(*FK*), there is another motion coupling factor requiring special consideration. Link 2 has three DOFs having been proved in the first loop, as shown in Eq. (31). Another constraint is brought to link 2 and its rotation around *X*-axis is restricted after the second loop is reclosed. Meanwhile, link 2 will affect link 4 through the kinematic pair *K*, and limit the rotation around *X*-axis of link 4 as well. In addition, having the same translation freedom along *Z*-axis as link 2, link 4 still has another translation freedom along *X*-axis. Therefore, link 4 has two DOFs after the second loop is reclosed. In this way, the screw system of virtual-loop chain which is composed of the screws of link 4 and kinematic pairs *D* and *E* is

$$\begin{aligned}
\$_{FK_1} &= (0 \quad 0 \quad 0; \quad 1 \quad 0 \quad 0), \\
\$_{FK_2} &= (0 \quad 0 \quad 0; \quad 0 \quad 0 \quad 1), \\
\$_{E_1} &= (0 \quad 0 \quad 1; \quad d_4 \quad e_4 \quad 0), \\
\$_{E_2} &= (0 \quad 0 \quad 0; \quad 0 \quad 0 \quad 1).
\end{aligned} \tag{34}$$

$E_1E_2FK_1FK_2$ in called a “virtual-loop mechanism”, as shown in Fig. 7, adopting the four screws a virtual loop is formed and over-constraint takes place at this loop reclosing moment. The frame in this loop is virtual.

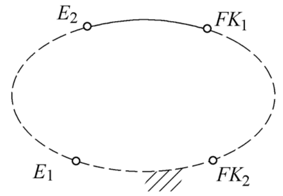

Fig. 7. Virtual closed-loop mechanism $E_1E_2FK_1FK_2$

According to the screw system of this virtual-loop mechanism, as shown in Eq. (34), it is easy to find its rank being three. That is, it has three reciprocal screws corresponding to three over-constraints, $\mu_3 = 3$. After the third loop is reclosed, it is clear that link 4 cannot translate along *X*-axis freely and it has only one DOF.

4.4 Sub-mechanism *GFHIG*

Because point *H* is a connecting point of the two mobile links, *IH* and link 4(*FK*), similarly, their screws should be determined firstly. For chain *IH*, it is a serial chain and has two kinematic pairs with three screws. Considering the mobility of another link 4(*FK*) and the motion coupling, it has only one DOF after the third loop is reclosed. Then the screw system of the virtual-loop chain including link 4 and kinematic pairs *H and I* has four screws:

$$\begin{aligned}
\$_{FK} &= (0 \quad 0 \quad 0; \quad 0 \quad 0 \quad 1), \\
\$_{H1} &= (0 \quad 0 \quad 0; \quad 0 \quad 0 \quad 1), \\
\$_{H2} &= (0 \quad 0 \quad 1; \quad d_3 \quad e_3 \quad 0), \\
\$_{I} &= (0 \quad 0 \quad 1; \quad d_4 \quad e_4 \quad 0),
\end{aligned} \tag{35}$$

where the four screws are linearly dependent and they have three reciprocal screws which denote three over-constraints, i.e. $\mu_4 = 3$.

4.5 Sub-mechanism *ABJIA*

Since point *J* connects the two moving links, *JIH* and link 2. For chain *IH*, because of the symmetry in size, point *J* of link *JIH* should just be located on point *J* of link 2 after the fourth loop is closed. Thus point *J* can also be treated as a fixed point. Next step, consider link 2, due to the strong motion coupling which exists among these loops, link 2 is constrained by each closure and has only one DOF right now. A virtual-loop chain composed of link 2 and joint *J* is obtained, the screw system can be expressed only with the following three screws:

$$\begin{aligned}
\$_{BK} &= (0 \quad 0 \quad 0; \quad 0 \quad 0 \quad 1), \\
\$_{J1} &= (0 \quad 0 \quad 0; \quad 0 \quad 0 \quad 1), \\
\$_{J2} &= (0 \quad 0 \quad 1; \quad d \quad e \quad 0),
\end{aligned} \tag{36}$$

which denotes the rank of the screw system is two, and therefore there are four reciprocal screws to the mechanism, and four over-constraints appear, $\mu_5 = 4$.

It can be learnt that just the strong motion coupling in this multi-loop mechanism has intensively reduced the mobility of link *BK* from three to one with every reclosing.

4.6 Whole mechanism

For the whole mechanism, $n=7$, $g=11$, the total number of the over-constraints is $\mu=\mu_1+\mu_2+\mu_3+\mu_4+\mu_5=1+1+3+3+4=12$. According to Eq. (1),

$$M=6(n-g-1)+\sum f_i+\mu=6(7-11-1)+19+12=1. \tag{37}$$

5 Conclusions

(1) The mobility analysis of the general multi-loop spatial mechanisms with strong motion couple is extremely complex and difficult. But the difficult mobility issue of the general multi-loop spatial mechanisms is firstly and perfectly solved completely.

(2) All of these kinds of the complex mechanisms are solvable, and difficult issues are transformed to be very easy by using the screw theory.

(3) The method to recognize over-constraints by analyzing relative movement is very clear in concept and easy to understand. And plus the method by virtual loop is simpler than the former.

(4) For multi-loop mechanisms, the motion couple and over-constraint all are affected by the structure of the loop-graph, and therefore the loop-graph has to be studied when mobility analysis is to determine whether or not there exists motion couple.

(5) This principle for the general multi-loop spatial mechanisms complements has enriched and developed the former universal mobility theory in a great way.

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