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General Analytical Shakedown Solution for Structures with Kinematic Hardening Materials

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Abstract: The effect of kinematic hardening behavior on the shakedown behaviors of structure has been investigated by performing shakedown analysis for some specific problems. The results obtained only show that the shakedown limit loads of structures with kinematic hardening model are larger than or equal to those with perfectly plastic model of the same initial yield stress. To further investigate the rules governing the different shakedown behaviors of kinematic hardening structures, the extended shakedown theorem for limited kinematic hardening is applied, the shakedown condition is then proposed, and a general analytical solution for the structural shakedown limit load is thus derived. The analytical shakedown limit loads for fully reversed cyclic loading and non-fully reversed cyclic loading are then given based on the general solution. The resulting analytical solution is applied to some specific problems: a hollow specimen subjected to tension and torsion, a flanged pipe subjected to pressure and axial force and a square plate with small central hole subjected to biaxial tension. The results obtained are compared with those in literatures, they are consistent with each other. Based on the resulting general analytical solution, rules governing the general effects of kinematic hardening behavior on the shakedown behavior of structure are clearly.

Keywords: elastic-plastic, elastic shakedown, kinematic hardening, shakedown limit load, general analytical solution

1 Introduction

Determining the structural shakedown behavior for structures under cyclic loading is important in structural evaluation and design. Generally, there are two different approaches to shakedown analysis, i.e., the static approach applying the static shakedown theorem and the kinematic approach applying the kinematic theorem. As the static approach results, in principle, in a conservative solution, it is more widely applied in engineering practices. The static shakedown theorem in its original formulation is valid only for elastic perfectly plastic(EPP) structures. However, in many engineering applications the materials exhibit kinematical hardening behavior, so kinematic hardening behavior needs to be taken into account to obtain more realistic results.

Already, the original static shakedown theorem has been extended to be suitable for structures with unlimited kinematical hardening materials. However, in real materials the stress is bounded by the ultimate stress, and the hardening is therefore limited. The limited kinematic hardening model is thus more realistic. The first explicit extended theorem for limited kinematic hardening materials was proposed by WEICHERT and GROß-WEEGE[1], employing a two-surface model. STEIN, et al[2–3], proposed another theorem employing an overlay model, which is actually an equivalent formulation of the former one. Even though WEICHERT and GROß-WEEGE only proved their theorem was valid for limited linear kinematical hardening, PHAM, et al[4–5], proved that in the generally-nonlinear case the shakedown theorem was valid for limited nonlinear kinematical hardening. Hence, the extended shakedown theorem for both the limited linear and nonlinear kinematic hardening can be expressed in the same formulation. As a consequence of the extended shakedown theorem, it was found that the specific type of kinematic hardening has little influence on the shakedown limit load; instead the initial yield stress and the ultimate stress are the main influence factors[4–7]. The extended shakedown theorem for both linear and nonlinear kinematic hardening mentioned above is referred to as the extended shakedown theorem for limited kinematic hardening in this paper.

Many shakedown analysis methods that take kinematic hardening behavior into account were developed, and solutions for the shakedown limit load were determined for some specified structures with kinematic hardening

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materials under specified loading conditions[8–10]. For example, shakedown behavior of a hollow tension specimen subjected to a constant tension and alternating torsion with zero mean shear stress was investigated with different material models, including the limited linear kinematic hardening[11], limited nonlinear kinematic hardening[12–13], limited generally-nonlinear kinematic hardening[14], unlimited kinematic hardening[14] and EPP[11, 14] models. Moreover, HEITZER, et al[11], derived the analytical shakedown limit load solutions for the same specimen with constant tension and cyclic torsion with nonzero mean value for both the limited linear and nonlinear kinematic hardening model. Additionally, the kinematic hardening model has also been applied to some other problems, such as a flange pipe subjected to thermo-mechanical cyclic loading[14–16], a square plate with small central hole subjected to biaxial cyclic tension[17–18], a grooved rectangular plate subjected to varying tension and bending[18], a cantilever hook subjected to a cyclic loading[19], a circular shaft subjected to cyclic torsion and a helical spring subjected to dynamic quasi-periodic fluctuating load[20], bar truss structures and bridge truss subjected to a cyclic force[21–22], a functionally graded rotating disk subjected to cyclic temperature gradient loading[23].

By comparing these specific solutions with the kinematic hardening material model and EPP model, rules governing specific effects can be drawn up; the kinematic hardening behavior has different effects on the shakedown behavior for different structures under different loading conditions. For example, the shakedown limit loads for the hollow tension specimen with the EPP material model are smaller than those with the kinematic hardening material model with the same initial yield stress. The shakedown limit loads of different kinematic hardening models with the same initial yield stress and ultimate stress differ little. Applying the same kinematic hardening model with the same initial yield stress has shown that the shakedown limit load is larger with larger ultimate stress[11–16]. Moreover, for a square plate with small central hole subjected to biaxial cyclic tension, the shakedown limit load is equal for the EPP and kinematic hardening models with the same initial yield stress[17]. These results have greatly helped our understanding of the effect of kinematic hardening behavior on structural shakedown behavior. However, the reasons for the different shakedown behaviors and conditions under which kinematic hardening can increase the shakedown limit load have not yet been fully explored, and further research is required. To resolve these problems rules governing the general effects of kinematic hardening behavior on the structural shakedown behavior are clearly necessary.

In this paper, the extended static shakedown theorem for limited kinematic hardening is used to derive the general analytical solution for the structural shakedown limit load. Then, rules governing the general effect of kinematic hardening behavior on the shakedown behavior of structures are proposed. The shakedown behavior is also investigated with the unlimited kinematic hardening and EPP models, which are specific cases of the limited kinematic hardening model; this is studied by setting the ultimate stress to infinite for the unlimited kinematic hardening model and equal to the initial yield stress for the EPP model. Furthermore, the analytical shakedown limit load for fully reversed cyclic loading with zero mean load value and non-fully reversed cyclic loading with non-zero mean value are given based on the general solution. The analytical results are applied to some specific problems and the solutions are compared with the solutions in the literature.

2 Static Shakedown Theorem

2.1 Classic shakedown theorem

The classic Melan's shakedown theorem for EPP materials can be formulated as follows.

If there exist a time-independent and self-equilibrium residual stress field $\boldsymbol{\sigma}_{ij}^{\mathrm{r}}(x)$, such that the yield condition $f_{\mathrm{Y}}(\boldsymbol{\sigma}_{ij}, \sigma_{\mathrm{Y}}] = 0$ is not violated for any loading path within the considered loading domain at any time t and in any point x of the structure, then the system will shake down The condition is expressed as

$$f_{\mathrm{Y}}[\boldsymbol{\sigma}^{\mathrm{e}}(x,t) + \boldsymbol{\sigma}_{ij}^{\mathrm{r}}(x), \sigma_{\mathrm{Y}}^{\mathrm{EPP}}] \leqslant 0, \tag{1}$$

where $\boldsymbol{\sigma}^{\mathrm{e}}(x,t)$ denotes the reference elastic stress field, which occurs in a fictitious, perfectly elastic reference body with the same geometric size under the same conditions as the elastic-plastic one, and $\sigma_{\mathrm{Y}}^{\mathrm{EPP}}$ is the yield stress of the material.

2.2 Theorem for kinematic hardening

The extended static shakedown theorem for unlimited kinematic hardening structures can be formulated as follows.

If there exist a time-independent and self-equilibrium residual stress field $\boldsymbol{\sigma}_{ij}^{\mathrm{r}}(x)$ and a time-independent back stress field $\boldsymbol{\alpha}_{ij}(x)$, such that the following condition is satisfied for any loading path within the considered loading domain at any time t and in any point x of the structure, then the system will shake down. The condition is expressed as

$$f_{\mathrm{Y}}[\boldsymbol{\sigma}_{ij}^{\mathrm{e}}(x,t) + \boldsymbol{\sigma}_{ij}^{\mathrm{r}}(x) - \boldsymbol{\alpha}_{ij}(x), \sigma_{\mathrm{Y}}] \leqslant 0, \tag{2}$$

where σ_{Y} is the initial yield stress of the kinematic hardening material.

WEICHERT and GROß-WEEGE introduced a bound surface $f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij}, \sigma_{\mathrm{H}}) = 0$ with respect to the ultimate stress to restrict the hardening. The extended static shakedown theorem for limited kinematical hardening can be formulated as follows.

If there exist a time-independent and self-equilibrium residual stress field $\boldsymbol{\sigma}_{ij}^{\mathrm{r}}(x)$ and a time-independent back stress field $\boldsymbol{\alpha}_{ij}(x)$, such that the yield condition $f_{\mathrm{Y}}(\boldsymbol{\sigma}_{ij},\sigma_{\mathrm{H}})=0$ and the bounding condition $f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij},\sigma_{\mathrm{H}})=0$ are not violated for any loading path within the considered loading domain at any time t and in any point x of the structure, then the system will shake down.

That is to say it needs to satisfy both inequality (2) and the following inequality:

$$f_{\mathrm{H}}[\boldsymbol{\sigma}_{ij}^{\mathrm{e}}(x,t)+\boldsymbol{\sigma}_{ij}^{\mathrm{r}}(x),\sigma_{\mathrm{H}}]\leqslant 0, \tag{3}$$

where σ_{H} is the ultimate stress of the kinematic hardening material. Melan's original theorem for perfectly plastic materials can also be deduced from the above formulation if $\boldsymbol{\alpha}_{ij}(x)=0$ and $\sigma_{\mathrm{H}}=\sigma_{\mathrm{Y}}$. Melan's theorem for unlimited kinematic hardening can also be deduced from the above formulation if $\sigma_{\mathrm{H}}\to\infty$. Thus inequality (3) is not relevant anymore for unlimited kinematic hardening.

3 Analytical Shakedown Analysis

3.1 Shakedown condition

A cyclic load $P(t)$ varies in the range $[P_{\mathrm{cmin}},P_{\mathrm{cmax}}]$, where P_{cmax} is the peak of the cyclic load and P_{cmin} is the valley of the cyclic load, while $(P_{\mathrm{cmax}}-P_{\mathrm{cmin}})$ is the scope of the cyclic load. For a kinematic hardening structure under cyclic load the structure can shake down if the requirements of the shakedown theorem are satisfied at both of the vertices of the cyclic load. So the shakedown conditions for structures based on Eq. (2) and (3) are expressed as follows:

$$f_{\mathrm{Y}}[\boldsymbol{\sigma}_{ij\max}^{\mathrm{e}}(x)+\boldsymbol{\sigma}_{ij}^{\mathrm{r}}(x)-\boldsymbol{\alpha}_{ij}(x),\sigma_{\mathrm{Y}}]\leqslant 0, \tag{4}$$

$$f_{\mathrm{Y}}[\boldsymbol{\sigma}_{ij\min}^{\mathrm{e}}(x)+\boldsymbol{\sigma}_{ij}^{\mathrm{r}}(x)-\boldsymbol{\alpha}_{ij}(x),\sigma_{\mathrm{Y}}]\leqslant 0, \tag{5}$$

$$f_{\mathrm{H}}[\boldsymbol{\sigma}_{ij\max}^{\mathrm{e}}(x)+\boldsymbol{\sigma}_{ij}^{\mathrm{r}}(x),\sigma_{\mathrm{H}}]\leqslant 0, \tag{6}$$

$$f_{\mathrm{H}}[\boldsymbol{\sigma}_{ij\min}^{\mathrm{e}}(x)+\boldsymbol{\sigma}_{ij}^{\mathrm{r}}(x),\sigma_{\mathrm{H}}]\leqslant 0, \tag{7}$$

where $\boldsymbol{\sigma}_{ij\max}^{\mathrm{e}}(x)$ is the reference elastic stress field to the peak cyclic load P_{cmax} and $\boldsymbol{\sigma}_{ij\min}^{\mathrm{e}}(x)$ is the reference elastic stress field to the valley cyclic load P_{cmin}. For shakedown analysis under the cyclic load $P_{\mathrm{c}}(t)$, it is reasonable to assume that when the cyclic load $P_{\mathrm{c}}(t)$ reaches its peak value, i.e., $P_{\mathrm{c}}(t)=P_{\mathrm{cmax}}$, the structure is in the elastic-plastic state. Then according to above theorem, a self-equilibrium stress field $\boldsymbol{\sigma}_{ij}^{\mathrm{r}}(x)$ is constructed as

$$\boldsymbol{\sigma}_{ij}^{\mathrm{r}}(x)=\boldsymbol{\sigma}_{ij}^{1}(x)-\boldsymbol{\sigma}_{ij\max}^{\mathrm{e}}(x), \tag{8}$$

where $\boldsymbol{\sigma}_{ij}^{1}(x)$ is the elastic-plastic stress field to the peak cyclic load, i.e., $P_{\mathrm{c\,max}}$. The back stress $\boldsymbol{\alpha}_{ij}^{1}(x)$ is selected as the time-independent back stress, which is the back stress to the peak cycle load. The combination of the reference elastic stress to the cyclic load and the stress $\boldsymbol{\sigma}_{ij}^{1}(x)$ should then satisfy Eqs. (4)–(7). The shakedown conditions for structures employing the stress given by Eq. (8) are then as follows:

$$f_{\mathrm{Y}}[\boldsymbol{\sigma}_{ij}^{1}(x)-\boldsymbol{\alpha}_{ij}^{1}(x),\sigma_{\mathrm{Y}}]\leqslant 0, \tag{9}$$

$$f_{\mathrm{Y}}[\boldsymbol{\sigma}_{ij}^{1}(x)-\boldsymbol{\sigma}_{ij}^{\mathrm{E}}(x)-\boldsymbol{\alpha}_{ij}^{1}(x),\sigma_{\mathrm{Y}}]\leqslant 0, \tag{10}$$

$$f_{\mathrm{H}}[\boldsymbol{\sigma}_{ij}^{1}(x),\sigma_{\mathrm{Y}}]\leqslant 0, \tag{11}$$

$$f_{\mathrm{H}}[\boldsymbol{\sigma}_{ij}^{1}(x)-\boldsymbol{\sigma}_{ij}^{\mathrm{E}}(x),\sigma_{\mathrm{H}}]\leqslant 0, \tag{12}$$

where $\boldsymbol{\sigma}_{ij}^{\mathrm{E}}(x)$ is the elastic-plastic stress field under the scope of the cyclic load, and where $\boldsymbol{\sigma}_{ij}^{\mathrm{E}}(x)=\boldsymbol{\sigma}_{ij\max}^{\mathrm{e}}(x)-\boldsymbol{\sigma}_{ij\min}^{\mathrm{e}}(x)$ applies.

Since the structure is assumed to be in the elastic-plastic state under P_{cmax}, Eq. (9) is naturally satisfied. Then the shakedown conditions reduce to Eqs. (10) and (11), and the shakedown limit load is the maximum load range that satisfies Eqs. (10) and (11).

The shakedown behavior of the structure is determined by the weakest critical point. For simplicity, only the situation for the structural critical point is discussed next. The shakedown conditions above are all converted to the conditions of the stress state of the critical point. The material is assumed to follow the von Mises criterion, and the subsequent yield surface $f_{\mathrm{Y}}(\boldsymbol{\sigma}_{ij},\sigma_{\mathrm{Y}})=0$ and the bounding surface $f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij},\sigma_{\mathrm{H}})=0$ of the kinematic hardening material are described by the von Mises criterion and are expressed as follows:

$$f_{\mathrm{Y}}(\boldsymbol{\sigma}_{ij},\sigma_{\mathrm{Y}})= \\ [(\boldsymbol{\sigma}_{ij}-\boldsymbol{\alpha}_{ij})(\boldsymbol{\sigma}_{ij}-\boldsymbol{\alpha}_{ij})-3(\sigma_{\mathrm{m}}-\alpha_{\mathrm{m}})^2]/2-\sigma_{\mathrm{Y}}^2/3=0, \tag{13}$$

$$f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij},\sigma_{\mathrm{H}})=\left(\boldsymbol{\sigma}_{ij}\boldsymbol{\sigma}_{ij}-3{\sigma_{\mathrm{m}}}^2\right)/2-\sigma_{\mathrm{H}}^2/3=0, \tag{14}$$

where $\boldsymbol{\sigma}_{ij}$ is the elastic-plastic stress, $\boldsymbol{\alpha}_{ij}$ is the back stress, $\boldsymbol{\sigma}'_{ij}$ and $\boldsymbol{\alpha}'_{ij}$ are the deviatoric stress tensors of $\boldsymbol{\sigma}_{ij}$ and $\boldsymbol{\alpha}_{ij}$ respectively, σ_{m} and α_{m} are the mean stresses of $\boldsymbol{\sigma}_{ij}$ and $\boldsymbol{\alpha}_{ij}$ respectively.

Substituting the left terms of Eqs. (9) and (10) into the expression of Eq. (13) yields

$$f_{\mathrm{Y}}(\boldsymbol{\sigma}_{ij}^{1},\sigma_{\mathrm{Y}})=[(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\alpha}_{ij}^{1})(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\alpha}_{ij}^{1})-3(\sigma_{\mathrm{m}}^{1}-\alpha_{\mathrm{m}}^{1})^{2}]/2-(\sigma_{\mathrm{Y}})^{2}/3, \tag{15}$$

$$f_{\mathrm{Y}}(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}}-\boldsymbol{\alpha}_{ij}^{1},\sigma_{\mathrm{Y}})=\{(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\alpha}_{ij}^{1})(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\alpha}_{ij}^{1})-3(\sigma_{\mathrm{m}}^{1}-\alpha_{\mathrm{m}}^{1})^{2}-[2(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\alpha}_{ij}^{1})-\boldsymbol{\sigma}_{ij}^{\mathrm{E}}]\boldsymbol{\sigma}_{ij}^{\mathrm{E}}+3[2(\sigma_{m}^{1}-\alpha_{m}^{1})-\sigma_{m}^{\mathrm{E}}]\sigma_{m}^{\mathrm{E}}\}/2-(\sigma_{\mathrm{Y}})^{2}/3, \tag{16}$$

respectively, where $\boldsymbol{\sigma}_{ij}^{1'}$, $\boldsymbol{\alpha}_{ij}^{1'}$ and $\boldsymbol{\sigma}_{ij}^{\mathrm{E}'}$ are the deviatoric stress tensors of $\boldsymbol{\sigma}_{ij}^{1}, \boldsymbol{\alpha}_{ij}^{1}$ and $\boldsymbol{\sigma}_{ij}^{E}$ respectively.

Substituting the left term of Eqs. (11) and (12) into Eq. (14) yields

$$f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij}^{1},\sigma_{\mathrm{H}})=[\boldsymbol{\sigma}_{ij}^{1}\boldsymbol{\sigma}_{ij}^{1}-3(\sigma_{\mathrm{m}}^{1})^{2}]/2-\sigma_{\mathrm{H}}^{2}/3, \tag{17}$$

$$f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}},\sigma_{H})=\left[\boldsymbol{\sigma}_{ij}^{1}\boldsymbol{\sigma}_{ij}^{1}-3(\sigma_{\mathrm{m}}^{1})^{2}-(2\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}})\boldsymbol{\sigma}_{ij}^{\mathrm{E}}+3(2\sigma_{\mathrm{m}}^{1}-\sigma_{\mathrm{m}}^{\mathrm{E}})\sigma_{\mathrm{m}}^{\mathrm{E}}\right]/2-(\sigma_{\mathrm{H}})^{2}/3, \tag{18}$$

respectively.

Because the structure is assumed to be in the elastic-plastic state under P_{cmax}, Eq. (9) concerning the critical point is thus reduced to $f_{\mathrm{Y}}[\boldsymbol{\sigma}_{ij}^{1}(x)-\boldsymbol{\alpha}_{ij}^{1}(x),\sigma_{\mathrm{Y}}]=0$. Substituting this equation and Eq. (15) into Eq. (16) yields

$$f_{\mathrm{Y}}(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}}-\boldsymbol{\alpha}_{ij}^{1},\sigma_{\mathrm{Y}})=-1/2[2(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\alpha}_{ij}^{1})-\boldsymbol{\sigma}_{ij}^{\mathrm{E}}]\sigma_{ij}^{\mathrm{E}}+3[2(\sigma_{\mathrm{m}}^{1}-\alpha_{\mathrm{m}}^{1})-\sigma_{\mathrm{m}}^{\mathrm{E}}]\sigma_{\mathrm{m}}^{\mathrm{E}}/2. \tag{19}$$

By substituting Eq. (17) into Eq. (18), we obtain

$$f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}},\sigma_{\mathrm{H}})=f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij}^{1},\sigma_{\mathrm{H}})-[2(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}})\boldsymbol{\sigma}_{ij}^{\mathrm{E}}+3(2\sigma_{\mathrm{m}}^{1}-\sigma_{\mathrm{m}}^{\mathrm{E}})\sigma_{\mathrm{m}}^{\mathrm{E}}]/2. \tag{20}$$

According to the generalized version of Hooke's law the following two equations hold:

$$\boldsymbol{\sigma}_{ij}^{\mathrm{e}}=2G\boldsymbol{\varepsilon}_{ij}^{\mathrm{e}}+\lambda\varepsilon_{\mathrm{v}}^{\mathrm{e}}\delta_{ij}, \tag{21}$$

$$\sigma_{\mathrm{m}}^{\mathrm{e}}=K\varepsilon_{\mathrm{v}}^{\mathrm{e}}, \tag{22}$$

where

$$\lambda=\mu E/[(1+\mu)(1-2\mu)], \tag{23}$$

$$K=E/3(1-2\mu), \tag{24}$$

$$G=E/2(1+\mu). \tag{25}$$

Where $\boldsymbol{\varepsilon}_{ij}^{\mathrm{e}}$ is the elastic strain corresponding to $\boldsymbol{\sigma}_{ij}^{\mathrm{e}}$, $\varepsilon_{\mathrm{v}}^{\mathrm{e}}$ is the corresponding volumetric strain and δ_{ij} is the unit tensor. If *i*=*j*, then $\delta_{ij}=1$. If *i*≠*j*, then $\delta_{ij}=0$. *E* is the elastic modulus, μ is the Poisson ratio and *G* is the shear modulus. Substituting Eqs. (21)–(25) into Eqs. (19) and (20) yields

$$f_{\mathrm{Y}}(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}}-\boldsymbol{\alpha}_{ij}^{1},\sigma_{\mathrm{Y}})=-G[2(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\alpha}_{ij}^{1})-\boldsymbol{\sigma}_{ij}^{\mathrm{E}}]\boldsymbol{\varepsilon}_{ij}^{\mathrm{E}}+G[2(\sigma_{\mathrm{m}}^{1}-\alpha_{\mathrm{m}}^{1})-\sigma_{\mathrm{m}}^{\mathrm{E}}]\varepsilon_{\mathrm{v}}^{\mathrm{E}}, \tag{26}$$

$$f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}},\sigma_{\mathrm{H}})=f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij}^{1},\sigma_{\mathrm{H}})-G[(2\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}})\boldsymbol{\varepsilon}_{ij}^{\mathrm{E}}+(2\sigma_{\mathrm{m}}^{1}-\sigma_{\mathrm{m}}^{\mathrm{E}})\varepsilon_{\mathrm{v}}^{\mathrm{E}}]. \tag{27}$$

According to the relation between the stress tensor and the deviatoric stress tensor we have

$$\boldsymbol{\sigma}_{ij}=\boldsymbol{\sigma}_{ij}{}'+\sigma_{m}\delta_{ij}, \tag{28}$$

$$\boldsymbol{\varepsilon}_{ij}^{\mathrm{e}}=\boldsymbol{\varepsilon}_{ij}^{\mathrm{e}\prime}+\varepsilon_{\mathrm{m}}^{\mathrm{e}}\delta_{ij}, \tag{29}$$

where $\boldsymbol{\varepsilon}_{ij}^{\mathrm{e}\prime}$ is the deviatoric tensor of $\boldsymbol{\varepsilon}_{ij}^{\mathrm{e}}$, $\varepsilon_{\mathrm{m}}^{\mathrm{e}}$ is the mean strain of $\boldsymbol{\varepsilon}_{ij}^{\mathrm{e}}$. Substituting Eqs. (28), (29) into Eqs. (26), (27) yields

$$f_{\mathrm{Y}}(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}}-\boldsymbol{\alpha}_{ij}^{1},\sigma_{\mathrm{Y}})=-G\left\{[2(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\alpha}_{ij}^{1})']-\boldsymbol{\sigma}_{ij}^{\mathrm{E}\prime}\right\}\boldsymbol{\varepsilon}_{ij}^{\mathrm{E}\prime}, \tag{30}$$

$$f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}},\sigma_{\mathrm{H}})=f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij}^{1},\sigma_{\mathrm{H}})-G(2\boldsymbol{\sigma}_{ij}^{1\prime}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}\prime})\boldsymbol{\varepsilon}_{ij}^{\mathrm{E}\prime}. \tag{31}$$

Substituting Eq. (30) and inequality (31) into inequalities (10) and (12), respectively, yields

$$f_{\mathrm{Y}}(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}}-\boldsymbol{\alpha}_{ij}^{1},\sigma_{\mathrm{Y}})=-G[2(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\alpha}_{ij}^{1})'-\boldsymbol{\sigma}_{ij}^{\mathrm{E}\prime}]\boldsymbol{\varepsilon}_{ij}^{\mathrm{E}\prime}\leqslant 0, \tag{32}$$

$$f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij}^{1}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}},\sigma_{\mathrm{H}})=f_{\mathrm{H}}(\boldsymbol{\sigma}_{ij}^{1},\sigma_{\mathrm{H}})-G(2\boldsymbol{\sigma}_{ij}^{1\prime}-\boldsymbol{\sigma}_{ij}^{\mathrm{E}\prime})\boldsymbol{\varepsilon}_{ij}^{\mathrm{E}\prime}\leqslant 0. \tag{33}$$

Therefore, the shakedown conditions for the limited kinematic hardening structure are converted to inequalities (11), (32) and (33), corresponding to inequalities (11), (10) and (12), respectively. For the unlimited kinematic hardening structure, it only needs to satisfy inequality (32), as the ultimate stress is infinite. For the EPP structure, it then needs to satisfy inequalities (11), (32) and (33) simultaneously with $\sigma_{\mathrm{H}}=\sigma_{\mathrm{Y}}$.

3.2 Shakedown limit load

The shakedown limit load is the maximum range of the cyclic load under which the structure can reach shakedown. The shakedown limit load is expressed as $[P_{\mathrm{cmin}}^{s},P_{\mathrm{cmax}}^{s}]$, where P_{cmax}^{s} and P_{cmin}^{s} are the peak and valley of the shakedown limit load, respectively, while $P_{\mathrm{cmax}}^{s}-P_{\mathrm{cmin}}^{s}$ is the scope of the shakedown limit load.

The stresses $\boldsymbol{\sigma}_{ij}$, $\boldsymbol{\sigma}_{ij}^{\mathrm{e}}$ and strain $\boldsymbol{\varepsilon}_{ij}^{\mathrm{e}}$ are represented in one coordinate system to describe the shakedown condition of inequality (32). Vectors are used to represent the states of stress and strain, which are illustrated in Fig. 1.

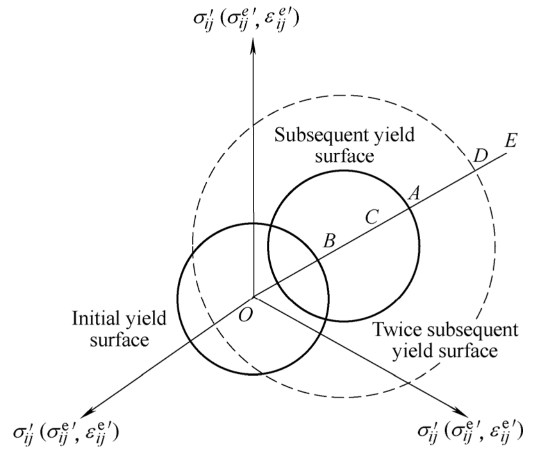

Fig. 1. Analytical analysis of inequality (32)

Setting $\boldsymbol{OA} = \boldsymbol{\sigma}^{1\prime}_{ij}$ and Point B to be the center of the subsequent yield surface, results in $\boldsymbol{OB} = \boldsymbol{\alpha}^{1\prime}_{ij}$ and $\boldsymbol{BA} = \boldsymbol{\sigma}^{1\prime}_{ij} - \boldsymbol{\alpha}^{1\prime}_{ij}$. Since $\boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij}$ and $\boldsymbol{\sigma}^{1\prime}_{ij} - \boldsymbol{\alpha}^{1\prime}_{ij}$ are in the same direction, then $\boldsymbol{BC}$ and $\boldsymbol{BA}$ are in the same direction if $\boldsymbol{BC} = \boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij}$ is set, which is illustrated in Fig. 1. Setting $\boldsymbol{BD} = 2(\boldsymbol{\sigma}^{1\prime}_{ij} - \boldsymbol{\alpha}^{1\prime}_{ij}) = 2\boldsymbol{BA}$ results in $\boldsymbol{CD} = 2(\boldsymbol{\sigma}^{1\prime}_{ij} - \boldsymbol{\alpha}^{1\prime}_{ij}) - \boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij}$. According to the generalized version of Hooke's law, $\boldsymbol{\varepsilon}^{\mathrm{E}\prime}_{ij} = \boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij} / 2G$, $G = E / 2(1+\mu)$, so $\boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij}$ and $\boldsymbol{\varepsilon}^{\mathrm{E}\prime}_{ij}$ are in the same direction. Setting $\boldsymbol{DE} = \boldsymbol{\varepsilon}^{\mathrm{E}\prime}_{ij}$, causes $\boldsymbol{DE}$ and $\boldsymbol{BC}$ to be in the same direction. Eq. (30) is thus expressed as

$$f_{\mathrm{Y}}(\boldsymbol{\sigma}^{1}_{ij} - \boldsymbol{\sigma}^{\mathrm{E}}_{ij} - \boldsymbol{\alpha}^{1}_{ij}, \sigma_{\mathrm{Y}}) = -G \bullet \boldsymbol{CD} \bullet \boldsymbol{DE}\,. \tag{34}$$

Eq. (34) can further be expressed as

$$f_{\mathrm{Y}}(\boldsymbol{\sigma}^{1}_{ij} - \boldsymbol{\sigma}^{\mathrm{E}}_{ij} - \boldsymbol{\alpha}^{1}_{ij}, \sigma_{\mathrm{Y}}) = -G \bullet \boldsymbol{CD} \bullet \boldsymbol{DE} \bullet \cos\psi\,, \tag{35}$$

where ψ is the angle between vectors $\boldsymbol{CD}$ and $\boldsymbol{DE}$. Since $G > 0$, the positive and negative of $f_{\mathrm{Y}}(\boldsymbol{\sigma}^{1}_{ij} - \boldsymbol{\sigma}^{\mathrm{E}}_{ij} - \boldsymbol{\alpha}^{1}_{ij}, \sigma_{\mathrm{Y}})$ is determined by the angle ψ between vectors $\boldsymbol{CD}$ and $\boldsymbol{DE}$. It is clear that if $|\boldsymbol{BC}| < |\boldsymbol{BD}|$, then the angle ψ is zero. So $f_{\mathrm{Y}}(\boldsymbol{\sigma}^{1}_{ij} - \boldsymbol{\sigma}^{\mathrm{E}}_{ij} - \boldsymbol{\alpha}^{1}_{ij}, \sigma_{\mathrm{Y}}) < 0$, and inequality (32) holds. If $|\boldsymbol{BC}| = |\boldsymbol{BD}|$, then $|\boldsymbol{CD}| = 0$ and $f_{\mathrm{Y}}(\boldsymbol{\sigma}^{1}_{ij} - \boldsymbol{\sigma}^{\mathrm{E}}_{ij} - \boldsymbol{\alpha}^{1}_{ij}, \sigma_{\mathrm{Y}}) = 0$, then inequality (32) holds too. If $|\boldsymbol{BC}| > |\boldsymbol{BD}|$, then the angle ψ is 180°. Therefore, $f_{\mathrm{Y}}(\boldsymbol{\sigma}^{1}_{ij} - \boldsymbol{\sigma}^{\mathrm{E}}_{ij} - \boldsymbol{\alpha}^{1}_{ij}, \sigma_{\mathrm{Y}}) > 0$, and inequality (32) does not hold anymore. The condition for which inequality (32) is true, is thus $|\boldsymbol{BC}| \leqslant |\boldsymbol{BD}|$, which is expressed in the form of stress

$$\left|\boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij}\right| \leqslant 2\left|\boldsymbol{\sigma}^{1\prime}_{ij} - \boldsymbol{\alpha}^{1\prime}_{ij}\right|, \tag{36}$$

where $\left|\boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij}\right|$ is the modulus of $\boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij}$, $\left|\boldsymbol{\sigma}^{1\prime}_{ij} - \boldsymbol{\alpha}^{1\prime}_{ij}\right|$ is the modulus of $\boldsymbol{\sigma}^{1\prime}_{ij} - \boldsymbol{\alpha}^{1\prime}_{ij}$.

Eq. (36) can be further expressed as

$$\left|\boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij}\right| \leqslant 2\delta_{\mathrm{Y}}\,, \tag{37}$$

where δ_{Y} is the radius of the subsequent yield surface. For kinematic hardening materials, δ_{Y} is constant. Therefore, for inequality (32) to hold the condition of external load is

$$P_{\mathrm{c\,max}} - P_{\mathrm{c\,min}} \leqslant 2P_{\mathrm{e}}\,, \tag{38}$$

where P_{e} is the elastic limit load of the structure for the critical point, and is referred to as the elastic limit load from hereon.

The condition under which inequality (33) holds true is discussed in the same way. As illustrated in Fig. 2, when $\boldsymbol{OA} = \boldsymbol{\sigma}^{1\prime}_{ij}, \boldsymbol{OC} = \boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij}$, and $\boldsymbol{OF} = 2\sigma^{1\prime}_{ij}$, then $\boldsymbol{CF} = 2\boldsymbol{\sigma}^{1\prime}_{ij} - \boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij}$. Setting $\boldsymbol{FE} = \varepsilon^{\mathrm{E}\prime}_{ij}$, inequality (31) is then expressed as

$$f_{\mathrm{H}}(\boldsymbol{\sigma}^{1}_{ij} - \boldsymbol{\sigma}^{\mathrm{E}}_{ij}, \sigma_{\mathrm{H}}) = f_{\mathrm{H}}(\boldsymbol{\sigma}^{1}_{ij}, \sigma_{\mathrm{H}}) - G \bullet \boldsymbol{CF} \bullet \boldsymbol{FE}\,. \tag{39}$$

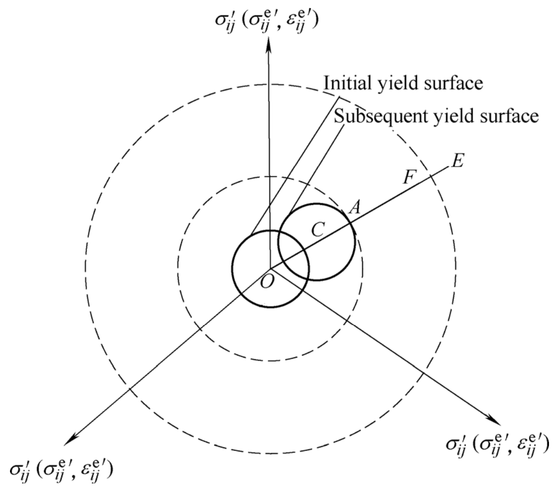

Fig. 2. Analytic analysis of inequality (33)

Similarly, if $|\boldsymbol{OC}| \leqslant |\boldsymbol{OF}|$, then the angle between vectors $\boldsymbol{CF}$ and $\boldsymbol{FE}$ is zero. In that case, $-G \bullet \boldsymbol{CF} \bullet \boldsymbol{FE} \leqslant 0$. If $|\boldsymbol{OC}| > |\boldsymbol{OF}|$ then the angle between vectors $\boldsymbol{CF}$ and $\boldsymbol{FE}$ is 180°. Therefore $-G \bullet \boldsymbol{CF} \bullet \boldsymbol{FE} > 0$ and the condition for inequality $-G(2\boldsymbol{\sigma}^{1\prime}_{ij} - \boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij})\boldsymbol{\varepsilon}^{\mathrm{E}\prime}_{ij} \leqslant 0$ to hold is $|\boldsymbol{OC}| \leqslant |\boldsymbol{OF}|$, which can be expressed in the form of stress

$$\left|\boldsymbol{\sigma}^{\mathrm{E}\prime}_{ij}\right| \leqslant 2\left|\boldsymbol{\sigma}^{1\prime}_{ij}\right|, \tag{40}$$

where $\left|\boldsymbol{\sigma}_{ij}^{1\prime}\right|$ is the modulus of $\boldsymbol{\sigma}_{ij}^{1\prime}$.

Because we have assumed that P_{cmax} exceeds the elastic limit load in the derivations of this paper, we also know that $\delta_Y \leqslant \left|\sigma_{ij}^{1\prime}\right|$. So if Eq. (37) holds, then Eq. (40) holds. If Eq. (11) also holds, then Eq. (33) holds too. That is to say, if the conditions for Eqs. (11) and (32) hold, then Eq. (33) holds too. The shakedown condition is reduced to Eqs. (11) and (32).

The corresponding external load to fulfill Eq. (11) should satisfy

$$P_{\mathrm{cmax}} \leqslant P_{\mathrm{H}}, \tag{41}$$

where P_{H} is the plastic limit load with respect to the ultimate stress σ_{H}.

The shakedown conditions of the external load are thus reduced to inequalities (38), and (41).

Therefore, the shakedown limit load for structures with limited kinematic hardening materials is the maximum load range that fulfills both inequalities (38) and (41).

It can be concluded from inequalities (38) and (41) that the shakedown behavior of the limited kinematic hardening structure is influenced by the elastic limit load and the plastic limit load with respect to the initial yield stress and the ultimate stress, respectively. For a general cyclic load, $P(t)$ varies in the range $[P_{\mathrm{cmin}}, P_{\mathrm{cmax}}]$, while the scope of its shakedown limit load, $P_{\mathrm{cmax}}^{\mathrm{s}} - P_{\mathrm{cmin}}^{\mathrm{s}}$, is bounded by twice the elastic limit load, and the peak of the shakedown limit load, $P_{\mathrm{cmax}}^{\mathrm{s}}$, is bounded by the plastic limit load corresponding to the ultimate stress.

For unlimited kinematic hardening, the bounding surface is no longer effective. Inequality (41) always holds. Therefore, the structure can shake down only if inequality (38) holds, i.e., if the scope of the cyclic load does not exceed twice the elastic limit load.

For EPP materials, the ultimate stress is equivalent to the yield stress, so the bounding condition is equivalent to the plastic limit condition with the yield stress of the EPP model. Therefore, the scope of the shakedown limit load is bounded by twice the elastic limit load, and the peak of the shakedown limit load is bounded by the plastic limit load with the yield stress of the EPP model.

3.3 Solutions for particular loading cases

The shakedown limit loads for fully reversed cyclic loading with zero mean value and cyclic loading with nonzero mean value are determined based on the above general analytical solution.

3.3.1 Fully reversed cyclic loading

If the cyclic load that varies in the range $P(t) \in [P_{\mathrm{cmin}}, P_{\mathrm{cmax}}]$ is fully reversed loading with zero mean value, then the peak and valley loads satisfy

$$P_{\mathrm{cmax}} = -P_{\mathrm{cmin}}. \tag{42}$$

Substituting that into inequality (38) yields

$$P_{\mathrm{cmax}} \leqslant P_{\mathrm{e}}. \tag{43}$$

For a limited kinematic hardening structure, the peak load should also satisfy inequality (41). Thus the peak of the shakedown limit load $P_{\mathrm{cmax}}^{\mathrm{s}}$ is the smaller one of P_{e} and P_{H}, i.e.

$$P_{\mathrm{cmax}}^{\mathrm{s}} = \min(P_{\mathrm{e}}, P_{\mathrm{H}}). \tag{44}$$

By substituting Eq.(44) into Eq.(42), the valley of the shakedown limit load $P_{\mathrm{cmin}}^{\mathrm{s}}$ is obtained as:

$$P_{\mathrm{cmin}}^{\mathrm{s}} = -\min(P_{\mathrm{e}}, P_{\mathrm{H}}). \tag{45}$$

For an unlimited kinematic hardening structure, only inequality (43) should be satisfied. Then the peak and valley of the shakedown limit load are

$$P_{\mathrm{cmax}}^{\mathrm{s}} = P_{\mathrm{e}}, \tag{46}$$

$$P_{\mathrm{cmin}}^{\mathrm{s}} = -P_{\mathrm{e}}. \tag{47}$$

For an EPP material, P_{H} is equivalent to the plastic limit load with yield stress, which is denoted as $P_{\mathrm{H}}^{\mathrm{EPP}}$. According to Eqs. (44) and (45), the shakedown limit loads of the peak and valley loads are as follows:

$$P_{\mathrm{cmax}}^{\mathrm{s}} = \min(P_{\mathrm{e}}, P_{\mathrm{H}}^{\mathrm{EPP}}), \tag{48}$$

$$P_{\mathrm{cmin}}^{\mathrm{s}} = -\min(P_{\mathrm{e}}, P_{\mathrm{H}}^{\mathrm{EPP}}). \tag{49}$$

3.3.2 Non-fully reversed cyclic loading

If the cyclic load that varies in the range $P(t) \in [P_{\mathrm{cmin}}, P_{\mathrm{cmax}}]$ is not fully reversed loading, but instead has a nonzero mean load P_{m}, then the peak and valley load can be expressed as follows

$$P_{\mathrm{max}} = (P_{\mathrm{max}} - P_{\mathrm{min}})/2 + P_m, \tag{50}$$

$$P_{\mathrm{min}} = -(P_{\mathrm{max}} - P_{\mathrm{min}})/2 + P_{\mathrm{m}}. \tag{51}$$

Substituting Eqs. (50) and (51) into inequality (38) yields

$$P_{\mathrm{cmax}} \leqslant P_{\mathrm{e}} + P_{\mathrm{m}}. \tag{52}$$

By combining inequality (52) with inequality (41), we find that the peak of the shakedown limit load $P_{\mathrm{cmax}}^{\mathrm{s}}$ for limited kinematic hardening is the smaller one of either $P_{\mathrm{e}} + P_{\mathrm{m}}$ or P_{H}, i.e.

$$P_{\mathrm{max}}^{\mathrm{s}} = \min(P_{\mathrm{e}} + P_{\mathrm{m}}, P_{\mathrm{H}}). \tag{53}$$

Then, with respect to inequality (38), the valley of the shakedown limit load P^{s}_{cmin} can be obtained as

$$P^{s}_{\min} = \min(P_e + P_m, P_H) - 2P_e \,. \tag{54}$$

For a special case, if $P_{\mathrm{cmin}} = 0$, i.e., the variation range of the cyclic load is $[0, P_{\mathrm{cmax}}]$, then the mean load must satisfy $P_m = P_{\mathrm{cmax}}/2$, and by substituting this into Eq. (53) the peak of the shakedown limit load is found to be

$$P^{s}_{\mathrm{cmax}} = \min(2P_e, P_H) \,. \tag{55}$$

For unlimited hardening, the peak of the shakedown limit load is

$$P^{s}_{\mathrm{cmax}} = 2P_e \,. \tag{56}$$

For an EPP material, P_H is equivalent to the plastic limit load P_H^{EPP} with respect to the yield stress. Therefore, according to Eq. (55) the peak of the shakedown limit load is

$$P^{s}_{\mathrm{cmax}} = \min(2P_e, P_H^{\mathrm{EPP}}) \,. \tag{57}$$

4 Discussion

The above analytical results are applied to some specific problems. The shakedown analysis results obtained are compared with the ones in the literature.

4.1 Hollow tension specimen

A hollow tension specimen is subjected to alternating torsion with zero mean shear stress such that $-\tau_{\max} = \tau_{\max}$ and a constant tensile stress $\sigma > 0$. The geometry of the specimen is illustrated in Fig. 3.

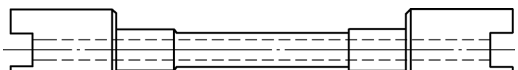

Fig. 3. Geometry of the hollow tension specimen

The analytical and numerical shakedown analyses are documented in the literature[11–14]. The normalized shakedown domain for the specimen is shown in Fig. 4. All the stress loads are normalized by the pure shakedown tension stress σ_s^{EPP} and the pure shakedown shear stress τ_s^{EPP} for the perfectly plastic material model.

For the EPP material model with only cyclic torsion acting on the specimen, it is clear that the elastic limit load is smaller than the plastic limit load, i.e., $\tau_e < \tau_H^{\mathrm{EPP}}$. According to Eq. (48), the peak of the shakedown limit load should satisfy $\tau^{s}_{\max} = \min(\tau_e, \tau_H^{\mathrm{EPP}})$; thus it can be obtained that $\tau^{s}_{\max} = \tau_e$. However, the relation between the plastic limit load, P_H, with respect to the ultimate stress, σ_H, and the limit load, P_H^{EPP}, with initial yield stress, σ_Y, is[11]

$$P_H = (\sigma_H / \sigma_Y) \bullet P_H^{\mathrm{EPP}} \,. \tag{58}$$

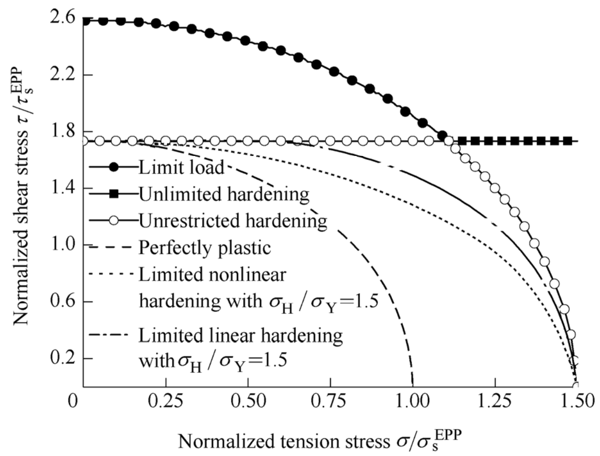

Fig. 4. Normalized shakedown domains of hollow specimen under constant tension and fully reversed alternating torsion[11–14]

Then the equality $\tau_e < (\sigma_H / \sigma_Y) \bullet \tau_H^{\mathrm{EPP}}$ holds, and so according to Eq. (44) the shakedown limit load of the torque for limited kinematic hardening is τ_e. For unlimited hardening, the shakedown limit load of the torque is τ_e according to Eq. (46). These results are consistent with the results presented on the vertical axis intercept in Fig. 4.

The hollow tension specimen illustrated in Fig. 3, which is subjected to constant moment, M and cyclic tension $N \in [0, N_{\max}]$ with nonzero mean value $N_{\max}/2$ was investigated by analytical and numerical methods[11]. The results are illustrated in Fig. 5.

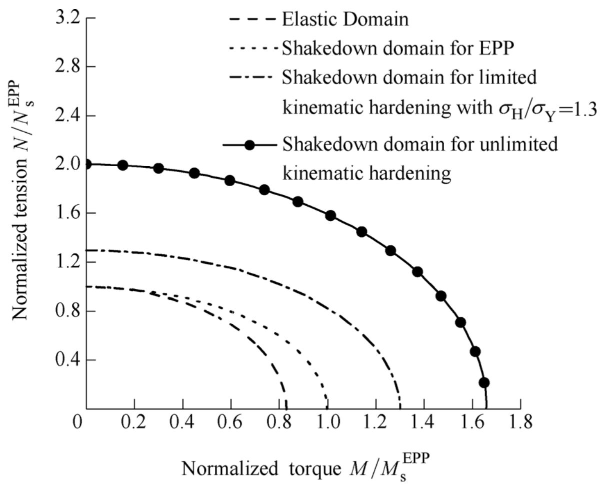

Fig. 5. Normalized shakedown domains of hollow specimen under constant torsion and non-fully reversed alternating tension[11]

For the hollow specimen with an EPP material model subjected only to tension, we know that the elastic limit load is equal to the plastic limit load, i.e., $N_H^{\mathrm{EPP}} = N_e$, owing to the homogeneous stress field in the specimen.

According to Eq. (57), the shakedown limit load for this loading case should satisfy $N_{\mathrm{s}}^{\mathrm{EPP}}=\min(2N_{\mathrm{e}},N_{\mathrm{H}}^{\mathrm{EPP}})$. Thus the shakedown limit load satisfies $N_{\mathrm{s}}^{\mathrm{EPP}}=N_{\mathrm{H}}^{\mathrm{EPP}}=N_{\mathrm{e}}<2N_{\mathrm{e}}$ in this case. For unlimited kinematic hardening, with respect to Eq. (56), the shakedown limit load of pure tension, N_{s}, is equal to $2N_{\mathrm{e}}$. Then, combining this with Eq. (55) and Eq. (58) under the condition $(\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}})<2$ we deduce that the shakedown limit load of pure tension for limited kinematic hardening must satisfy $N_{\mathrm{s}}=(\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}})N_{\mathrm{s}}^{\mathrm{EPP}}=(\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}})N_{\mathrm{e}}$, if $(\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}})\geqslant 2$, while the shakedown limit load of pure tension must satisfy $N_{\mathrm{s}}=2N_{\mathrm{e}}$. These results are consistent with the results illustrated on the vertical axis intercepts of Fig. 5, such as for EPP case, $N_{\mathrm{s}}^{\mathrm{EPP}}=N_{\mathrm{e}}$, for $\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}}=1.3$, $N_{\mathrm{s}}=1.3N_{\mathrm{s}}^{\mathrm{EPP}}$, for unlimited hardening $N_{\mathrm{s}}=2N_{\mathrm{e}}$.

When there is only constant torsion, the shakedown limit load is equal to the plastic limit load. Therefore, according to Eq. (58) we obtain that the shakedown limit load for limited kinematic hardening is a proportional enlargement of the perfectly plastic shakedown load limit by a factor of $\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}}$. For unlimited kinematic hardening, with respect to Eq. (56), the shakedown limit load of pure constant torque is equal to $2M_{\mathrm{e}}$. Thus we obtain that if $(\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}})<2$, then the shakedown limit load of pure torsion for limited kinematic hardening is $M_{\mathrm{s}}=(\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}})M_{\mathrm{H}}^{\mathrm{EPP}}=(\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}})M_{\mathrm{s}}^{\mathrm{EPP}}$, and if $(\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}})\geqslant 2$ then the shakedown limit load of pure torsion for limited kinematic hardening must satisfy $M_{\mathrm{s}}=2M_{\mathrm{e}}$. This is consistent with the results illustrated on the horizontal axis intercepts of Fig. 5, such as for the case when $\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}}=1.3$ and $M_{\mathrm{s}}=1.3M_{\mathrm{s}}^{\mathrm{EPP}}$, for unlimited hardening $M_{\mathrm{s}}=2M_{\mathrm{e}}$.

4.2 Flanged pipe

A flanged pipe is subjected to an internal pressure, p, and an axial force, Q, which vary independently in the ranges $p\in[0,p_{\max}]$ and $Q\in[0,Q_{\max}]$, respectively. The flanged pipe is illustrated in Fig. 6.

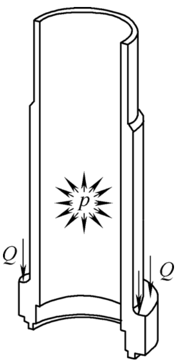

Fig. 6. Sketch and loadings of the flanged pipe[14]

The shakedown domain of a flanged pipe was investigated in Refs. [14–16]. The shakedown analysis results are presented in Fig. 7.

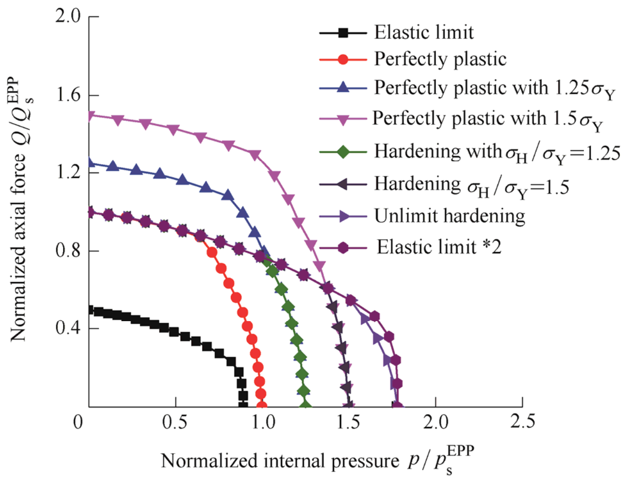

Fig. 7. Normalized shakedown domains of flanged pipe[14–16]

As shown in Fig. 7, for an EPP model with only cyclic internal pressure, the shakedown limit load is larger than the elastic limit load, p_e, and smaller than twice the elastic limit load $2p_e$. According to Eq. (57), $p_{\mathrm{s}}^{\mathrm{EPP}}=\min(2p_{\mathrm{e}},p_{\mathrm{H}}^{\mathrm{EPP}})$ should be satisfied. Thus $p_{\mathrm{s}}^{\mathrm{EPP}}=p_{\mathrm{H}}^{\mathrm{EPP}}<2P_{\mathrm{e}}$ holds in this case. For an unlimited kinematic hardening model, with respect to Eq. (56), the shakedown limit load of pure internal pressure is equal to $2p_{\mathrm{e}}$. Combining with Eqs. (55) and (58) we then obtain that if $p_{\mathrm{H}}=(\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}})p_{\mathrm{H}}^{\mathrm{EPP}}<2P_{\mathrm{e}}$, the shakedown limit load of pure pressure for limited kinematic hardening must satisfy $p_{\mathrm{s}}=(\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}})p_{\mathrm{s}}^{\mathrm{EPP}}$, while if $p_{\mathrm{H}}=(\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}})P_{\mathrm{H}}^{\mathrm{EPP}}\geqslant 2P_{\mathrm{e}}$, the shakedown limit load of pure pressure must satisfy $p_{\mathrm{s}}=2p_{\mathrm{e}}$. This is consistent with the results illustrated on the horizontal axis intercepts of Fig. 7, such as for the case in which $\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}}=1.25$, $p_{\mathrm{s}}=1.25p_{\mathrm{s}}^{\mathrm{EPP}}$, for $\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}}=1.5$, $p_{\mathrm{s}}=1.5p_{\mathrm{s}}^{\mathrm{EPP}}$, for unlimited hardening $p_{\mathrm{s}}=2p_{\mathrm{e}}$.

As shown in Fig. 7, for an EPP model with only cyclic axial force, Q, the shakedown limit load is equal to twice the elastic limit load, Q_{e}, i.e., $Q_{\mathrm{s}}^{\mathrm{EPP}}=2Q_{\mathrm{e}}$. According to Eq. (57), $Q_{\mathrm{s}}^{\mathrm{EPP}}=\min(2Q_{\mathrm{e}},Q_{\mathrm{H}}^{\mathrm{EPP}})$ must be satisfied. This therefore demonstrates that $Q_{\mathrm{s}}^{\mathrm{EPP}}=2Q_{\mathrm{e}}<Q_{\mathrm{H}}^{\mathrm{EPP}}$ holds in this case. Further, when we combine the above with Eq. (58) we obtain that the limit load Q_{H} satisfies $Q_{\mathrm{H}}=(\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}})Q_{\mathrm{H}}^{\mathrm{EPP}}>Q_{\mathrm{H}}^{\mathrm{EPP}}>2Q_{\mathrm{e}}$. Then, according to Eq. (55), the shakedown limit load of pure axial force must satisfy $Q_{\mathrm{s}}=2Q_{\mathrm{e}}$ for the limited kinematic hardening model. For an unlimited kinematic hardening material model, with respect to Eq. (56), the shakedown limit load of axial force is also equal to $2Q_{\mathrm{e}}$. These results coincide with the ones illustrated on the vertical axis intercepts of Fig. 7, such as for the case of $\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}}=1.25$, $\sigma_{\mathrm{H}}/\sigma_{\mathrm{Y}}=1.5$ and for unlimited hardening, when the shakedown limit load of pure axial force always satisfies $Q_{\mathrm{s}}=2Q_{\mathrm{e}}$.

4.3 Square plate with a central hole

The square plate with a small central hole is subjected to a system of uniform horizontal and vertical tensile stresses F_1 and F_2, respectively. The two stress loads vary cyclically in the ranges $F_1 \in [0, F_1^{\max}]$ and $F_2 \in [0, F_2^{\max}]$, respectively. The plate is illustrated in Fig. 8.

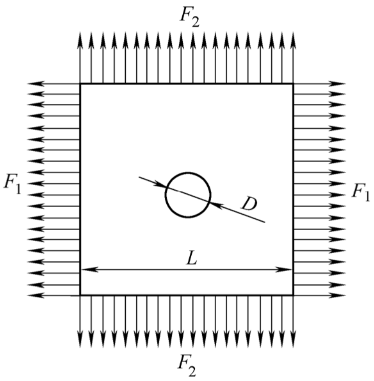

Fig. 8. Sketch and loadings of square plate with a central hole

The shakedown domains of the square plate for an EPP and unlimited linear kinematic hardening model, in which the initial yield stress was the same as the yield stress for the EPP model, were investigated by ABDALLA, et al[17]. The shakedown analysis results are presented in Fig. 9.

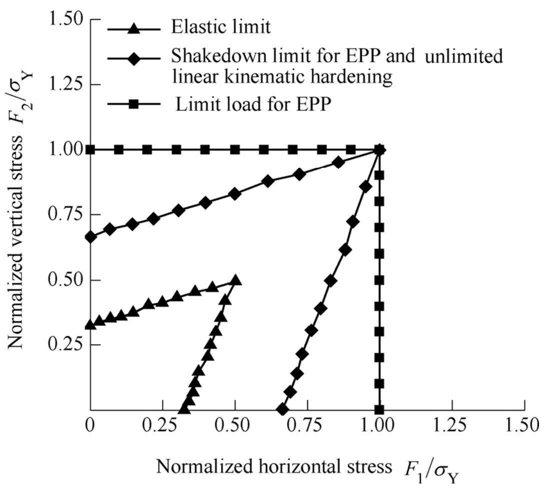

Fig. 9. Normalized load domains of square plate[17]

As shown in Fig. 9, for an EPP model with only horizontal or vertical cyclic tension stress, twice the elastic limit load is less than the plastic limit load when $\sigma_H = \sigma_Y$, i.e., $2F_e < F_H^{EPP}$. And according to Eq. (57), the shakedown limit load must satisfy $F_s^{EPP} = \min(2F_e, F_H^{EPP})$ so that $F_s^{EPP} = 2F_e$ holds in this case. For unlimited kinematic hardening materials, with respect to Eq. (56), the shakedown limit load of the tension stress is also equal to $2F_e$. In this case the shakedown limit load for the unlimited kinematic hardening and EPP models are the same. These results agree with the ones illustrated on the intercepts of the two axes of Fig. 9, as Fig. 9 shows that for EPP and unlimited hardening, the shakedown limit load must always satisfy $F_s = 2F_e$.

5 Conclusions

(1) For a structure with limited kinematic hardening material, the shakedown limit load is correlated to the initial yield stress and the ultimate stress. The scope of the shakedown limit load is bounded by twice the elastic limit load, and the peak of the shakedown limit load is bounded by the plastic limit load with respect to the ultimate stress.

(2) A structure with unlimited kinematic hardening material can reach shakedown only if the scope of the load does not exceed twice the elastic limit load.

(3) For a structure with EPP material, the scope of the shakedown limit load is bounded by twice the elastic limit load, and the peak of the shakedown limit load is bounded by the plastic limit load with respect to the yield stress.

(4) The results for some specific problems that are obtained by applying the analytical solutions are consistent with the results in the literature for those same problems.

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